A DEFORMATION FOR HOMOTOPY FIXED POINT SPECTRAL SEQUENCES

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Abstract. In this paper, we modify the construction of synthetic spectra and introduce a sheaf-theoretical machine that produces deformations. Given a well-behaved stable ∞-category $\mathcal{C}$, this machine constructs a deformation of which the generic fiber is equivalent to the given category $\mathcal{C}$.

In particular, for a finite group G, we apply this machine to the ∞-category of genuine G-spectra $\mathcal{S}p_G$ and introduce the ∞-category of $\mathbb{S}[G]$-synthetic spectra $\mathcal{S}yn_{\mathbb{S}[G]}$, which serves as a categorification of the $\mathbb{Z}$-graded, Mackey-functor-valued homotopy fixed point spectral sequences in $\mathcal{S}p_G$. Moreover, we state a comparison between $\mathbb{S}[G]$-synthetic spectra and filtered G-spectra through an adjoint pair and prove that the right adjoint sends the synthetic analogue of a G-spectrum to the décalage of its homotopy fixed point filtration.

Contents

1. Introduction

The ∞-categorical deformation theory provides new calculation tools in the stable homotopy theory. For instance, in [Pst23], Pstrągowski introduces the ∞-category of *E-synthetic spectra* $\mathcal{S}yn_E$ for a homotopy ring spectrum E of Adams-type, which can be viewed as a deformation with the generic fiber being the stable ∞-category of spectra $\mathcal{S}p$. The category of E-synthetic spectra $\mathcal{S}yn_E$ can also be regarded as a categorification of E-based Adams spectral sequences in $\mathcal{S}p$.

The theory of synthetic spectra produces useful computational techniques for Adams spectral sequences. Using $\mathbb{F}_2$-synthetic spectra, Lin, Wang, and Xu [LWX25] develop the Generalized Leibniz Rule and the Generalized Mahowald Trick. They further applied these two methods, together with Lin's computer programs and several ad hoc arguments, to prove that the element h_6^2 is a permanent cycle in the $\mathbb{F}_2$-Adams spectral sequence of the sphere spectrum. This further implies the existence of framed manifolds of Kervaire invariant one in dimension 126, resolving the last case of the Kervaire invariant problem.

The category $\mathcal{S}yn_E$ of E-synthetic spectra can be interpreted in two ways: as the category of modules in $\mathrm{Fil}(\mathcal{S}p)$, the category of filtered spectra, over a certain filtered ring spectra, or as the category of spherical sheaves of spectra over a certain ∞-site. Both descriptions have their own advantages. The filtered spectra viewpoint is closely related to spectral sequences, while the sheaf-theoretical viewpoint provides more structure: the natural t-structure on $\mathcal{S}yn_E$, which leads to more computational tools.

The main goal of this paper is to generalize the theory of synthetic spectra to equivariant homotopy theory. This problem has been studied in [AKBBK23], in which Angelini-Knoll, Behrens, Belmont, and Kong construct a deformation category with the generic fiber being the category of *Borel G-spectra* for a finite group G from the filtered objects point of view. In particular, this deformation categorifies the *E modified Adams spectral sequences* of Borel G-spectra.

As mentioned above, to deduce more computational tools, we also expect to find a sheaf-theoretical construction in the equivariant homotopy theory. In this paper, we generalize the construction of synthetic spectra and introduce a sheaf-theoretical machine. Given a stable ∞-category $\mathcal{C}$ that satisfies Assumptions 2.1, this machine produces a deformation of which the generic fiber is equivalent to the given category $\mathcal{C}$. In particular, applying this machine, we define a deformation $\mathcal{S}yn_{\mathbb{S}[G]}$, called the category of $\mathbb{S}[G]$-synthetic spectra, with the generic fiber being the ∞-category $\mathcal{S}p_G$ of genuine G-spectra.

In contrast to [AKBBK23], this paper works in genuine G-spectra, and the deformation $\mathcal{S}yn_{\mathbb{S}[G]}$ is constructed sheaf-theoretically. Moreover, we are focusing on the case where $E = \mathbb{S}$, the sphere spectrum. The deformation $\mathcal{S}yn_{\mathbb{S}[G]}$ categorifies the $\mathbb{S}$ modified Adams spectral sequences, which coincide with the homotopy fixed point spectral sequences in $\mathcal{S}p_G$. The same argument also works for E modified Adams spectral sequences when E is a compact ring spectrum, but would fail in the non-compact case.

In analogy with synthetic spectra, we expect that the theory of $\mathbb{S}[G]$-synthetic spectra would provide new computational techniques for the G-homotopy fixed point spectral sequences.

1.1. **∞-categorical deformation theory.** We first introduce some general features of the ∞-categorical deformation theory, mainly following [Bar23].

To begin with, let $\mathrm{Fil}(\mathcal{S}p) \simeq \mathrm{Fun}((\mathbb{Z}^{\leqslant})^{\mathrm{op}}, \mathcal{S}p)$ be the ∞-category of filtered spectra. This category is stable and presentably symmetric monoidal through Day convolution. Let $\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}$ be the ∞-category of presentable stable ∞-categories and left adjoints, which admits a symmetric monoidal structure given by Lurie's tensor product [Lura, Section 4.8]. Then, $\mathrm{Fil}(\mathcal{S}p)$ admits an $\mathbb{E}_\infty$-algebra structure in $\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}$.

Definition 1.1. A *one-parameter deformation* is a $\mathrm{Fil}(\mathcal{S}p)$-module $\mathcal{C}^{\mathrm{def}}$ in $\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}$. Moreover, given an ∞-operad $\mathcal{O}$, an *$\mathcal{O}$-monoidal one-parameter deformation* is an $\mathcal{O}$-algebra in $\mathcal{M}od_{\mathrm{Fil}(\mathcal{S}p)}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$.

Convention 1.2. We are more interested in *$\mathbb{E}_\infty$-monoidal one-parameter deformations*, so we abbreviate the notation and simply say $\mathcal{C}^{\mathrm{def}}$ is a *deformation* if it is an $\mathbb{E}_\infty$-monoidal one-parameter deformation; that is, a $\mathrm{Fil}(\mathcal{S}p)$-algebra in $\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}$.

Remark 1.3. By definition, $\mathrm{Fil}(\mathcal{S}p)$ is the category of presheaves of spectra on the poset of integers $\mathbb{Z}^{\leqslant}$. Then, for any presentably symmetric monoidal stable ∞-category $\mathcal{C}^{\mathrm{def}} \in \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$, a $\mathrm{Fil}(\mathcal{S}p)$-algebra structure on $\mathcal{C}^{\mathrm{def}}$ is equivalently a symmetric monoidal functor from the poset $\mathbb{Z}^{\leqslant}$ to $\mathcal{C}^{\mathrm{def}}$.

According to Example A.5, $\mathrm{Fil}(\mathcal{S}p)$ admits a (smashing) bi-grading $\Sigma^{a,b}$ and a natural transformation $\tau : \Sigma^{0,-1} \to id_{\mathrm{Fil}(\mathcal{S}p)}$ given by

$$\begin{tikzcd}
\Sigma^{0,-1}X : \arrow[d, "\tau_X"] & \cdots \arrow[r] & X_{n+2} \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & \cdots \\
X : & \cdots \arrow[r] & X_{n+1} \arrow[r] & X_n \arrow[r] & X_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 1.4. Given a one-parameter deformation $\mathcal{C}^{\mathrm{def}} \in \mathcal{M}od_{\mathrm{Fil}(\mathcal{S}p)}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$, we introduce a bi-grading

$$\begin{aligned}\Sigma^{a,b} : \mathcal{C}^{\mathrm{def}} &\to \mathcal{C}^{\mathrm{def}} \\ M &\mapsto \Sigma^{a,b}\mathbb{1}_{\mathrm{Fil}(\mathcal{S}p)} \otimes M\end{aligned}$$

and a natural transformation

$$\tau : \Sigma^{0,-1} \to id \in \mathrm{Fun}(\mathcal{C}^{\mathrm{def}}, \mathcal{C}^{\mathrm{def}})$$

with $\tau_M \simeq \tau_{\mathbb{1}} \otimes M : \Sigma^{0,-1}M \to M$. We call the natural transformation τ on $\mathcal{C}^{\mathrm{def}}$ *the deformation parameter*.

Now we consider two symmetric monoidal left adjoints: the *underlying spectra functor*

$$\mathrm{colim} : \mathrm{Fil}(\mathcal{S}p) \to \mathcal{S}p$$

and the *associated graded functor*

$$\mathrm{Gr} : \mathrm{Fil}(\mathcal{S}p) \to \mathrm{Gr}(\mathcal{S}p).$$

Definition 1.5. Let $\mathcal{C}^{\mathrm{def}} \in \mathcal{M}od_{\mathrm{Fil}(\mathcal{S}p)}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$ be a one-parameter deformation. The *generic fiber* of $\mathcal{C}^{\mathrm{def}}$ is the base-change along the underlying spectra functor

$$\mathcal{S}p \otimes_{\mathrm{Fil}(\mathcal{S}p)} \mathcal{C}^{\mathrm{def}} \in \mathcal{M}od_{\mathcal{S}p}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}) \simeq \mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}.$$

The *special fiber* of $\mathcal{C}^{\mathrm{def}}$ is the base-change along the associated graded functor

$$\mathrm{Gr}(\mathcal{S}p) \otimes_{\mathrm{Fil}(\mathcal{S}p)} \mathcal{C}^{\mathrm{def}} \in \mathcal{M}od_{\mathrm{Gr}(\mathcal{S}p)}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}).$$

Warning 1.6. Our definition of special fiber is slightly different from [Bar23]. Barkan defines the special fiber as the base-change along

$$\mathrm{Fil}(\mathcal{S}p) \xrightarrow{\mathrm{Gr}} \mathrm{Gr}(\mathcal{S}p) \xrightarrow{\oplus} \mathcal{S}p$$
$$(X_n)_n \mapsto \bigoplus_n X_n$$

Remark 1.7. Let $\mathcal{C}^{\mathrm{def}} \in \mathcal{M}od_{\mathrm{Fil}(\mathcal{S}p)}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$ be a one-parameter deformation. Note that the underlying spectra functor

$$\mathrm{colim} : \mathrm{Fil}(\mathcal{S}p) \to \mathcal{S}p$$

exhibits $\mathcal{S}p$ as the localization on the strongly saturated collection of morphisms generated by

$$\{\tau_X : \Sigma^{0,-1}X \to X\}_{X \in \mathrm{Fil}(\mathcal{S}p).}$$

Then, the generic fiber of $\mathcal{C}^{\mathrm{def}}$ is precisely the localization $\mathcal{C}^{\mathrm{def}}[\tau_M^{-1}]$ on the strongly saturated collection of morphisms generated by

$$\{\tau_M : \Sigma^{0,-1}M \to M\}_{M \in \mathcal{C}^{\mathrm{def}}},$$

where τ is the deformation parameter.

Similarly, note that the associated graded functor

$$\mathrm{Gr} : \mathrm{Fil}(\mathcal{S}p) \to \mathrm{Gr}(\mathcal{S}p)$$

exhibits $\mathrm{Gr}(\mathcal{S}p)$ as the category of modules over $C\tau$ in $\mathrm{Fil}(\mathcal{S}p)$, where $C\tau$ is the image of the unit in $\mathrm{Gr}(\mathcal{S}p)$ under the right adjoint to Gr, which is, in particular, an $\mathbb{E}_\infty$-algebra in $\mathrm{Fil}(\mathcal{S}p)$. Then, tensoring with $C\tau$ defines a monad on $\mathcal{C}^{\mathrm{def}}$, and the special fiber of $\mathcal{C}^{\mathrm{def}}$ is precisely the category of modules over the modad $C\tau$.

In particular, if $\mathcal{C}^{\mathrm{def}} \in \mathrm{CAlg}_{\mathrm{Fil}(\mathcal{S}p)}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$ is an $\mathbb{E}_\infty$-monoidal one-parameter deformation, then the unit $i : \mathrm{Fil}(\mathcal{S}p) \to \mathcal{C}^{\mathrm{def}}$ sends $C\tau \in \mathrm{CAlg}(\mathrm{Fil}(\mathcal{S}p))$ to an $\mathbb{E}_\infty$-algebra in $\mathcal{C}^{\mathrm{def}}$, and the special fiber of $\mathcal{C}^{\mathrm{def}}$ is the category of modules over $i(C\tau)$.

Philosophically, we would expect that a deformation $\mathcal{C}^{\mathrm{def}}$ should encode the information of a certain type of spectral sequences defined in the generic fiber

$$\mathcal{C} \simeq \mathcal{S}p \otimes_{\mathrm{Fil}(\mathcal{S}p)} \mathcal{C}^{\mathrm{def}}.$$

To see this, let $\nu : \mathcal{C} \to \mathcal{C}^{\mathrm{def}}$ be a suitable section of $\tau^{-1} : \mathcal{C}^{\mathrm{def}} \to \mathcal{C}$. Here, we do not require ν to be a morphism in $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$, but merely a functor between ∞-categories.

Recall that by [Hei23], the $\mathrm{Fil}(\mathcal{S}p)$-module $\mathcal{C}^{\mathrm{def}}$ is canonically enriched in $\mathrm{Fil}(\mathcal{S}p)$. Then for each pair of objects $X, Y \in \mathcal{C}$, the underlying spectrum of the internal mapping filtered spectrum

$$\mathrm{F}_*(\nu(X), \nu(Y)) \in \mathrm{Fil}(\mathcal{S}p)$$

is the mapping spectra $\mathrm{F}(X, Y)$ between X and Y in $\mathcal{C}$ because ν is a section of τ^{-1}.

By [Lura, 1.2.2], the filtered spectrum $\mathrm{F}_*(\nu(X), \nu(Y))$ further induces a spectral sequence that conditionally converges to the homotopy group of (a suitable completion of) $\mathrm{F}(X, Y) \in \mathcal{S}p$.

Therefore, if we begin with a stable ∞-category $\mathcal{C}$ and already have defined a collection of spectral sequences on each mapping spectra in $\mathcal{C}$ in a functorial way, then we expect to construct a deformation $\mathcal{C}^{\mathrm{def}}$ of which the generic fiber is equivalent to $\mathcal{C}$, together with a section ν, to categorify the collection of spectral sequences as above. Some important examples of deformations arise in this way.

For example, consider the *motivic homotopy theory* introduced by Morel and Voevodsky (e.g. [MV99]). Over the field of complex numbers $\mathbb{C}$, we have the *Betti realization* functor

$$Re : \mathcal{S}p_{\mathbb{C}} \to \mathcal{S}p$$

from the category of cellular $\mathbb{C}$-motivic spectra $\mathcal{S}p_{\mathbb{C}}$ to the category of classical spectra $\mathcal{S}p$. In [Lev15], Levine proves that the Betti realization functor sends the motivic slice spectral sequences in $\mathcal{S}p_{\mathbb{C}}$ to the Adams-Novikov spectral sequences in $\mathcal{S}p$ after re-grading.

This result could be reinterpreted as follows. Let $MU \in \mathcal{S}p$ be the *complex cobordism spectrum* and consider the *even cosimplicial décalage* of the cobar complex of MU

$$\Gamma_*(\mathbb{S}) := \operatorname{Tot} \tau_{\geqslant 2*}(MU^{\otimes \bullet+1}) \in \operatorname{Fil}(\mathcal{S}p).$$

Since MU is an $\mathbb{E}_\infty$-algebra in $\mathcal{S}p$, $\Gamma_*(\mathbb{S})$ admits an $\mathbb{E}_\infty$-algebra structure in $\operatorname{Fil}(\mathcal{S}p)$, so we can define the category of $\Gamma_*(\mathbb{S})$-modules

$$\mathcal{M}od_{\Gamma_*(\mathbb{S})}(\operatorname{Fil}(\mathcal{S}p))$$

which is automatically a $\operatorname{Fil}(\mathcal{S}p)$-algebra.

Surprisingly, for each prime p, the category of cellular $\mathbb{C}$-motivic spectra $\mathcal{S}p_{\mathbb{C}}$ coincides with $\mathcal{M}od_{\Gamma_*(\mathbb{S})}(\operatorname{Fil}(\mathcal{S}p))$ after p-completion, and Betti realization coincides with the underlying spectra functor. A proof of this at prime $p = 2$ could be found in [GIKR22]. Note that the underlying spectra of $\Gamma_*(\mathbb{S})$ is the MU-nilpotent completion of the sphere spectrum $\mathbb{S}$, which is just $\mathbb{S}$ itself since the sphere spectrum is connective. Hence, the category of p-complete cellular $\mathbb{C}$-motivic spectra $(\mathcal{S}p_{\mathbb{C}})_p^\wedge$ is a deformation of which the generic fiber is equivalent to the ∞-category of p-complete spectra $\mathcal{S}p_p^\wedge$. This provides the first interesting example of deformations.

In general, given a presentably symmetric monoidal stable ∞-category $\mathcal{C}$, to construct a deformation, we can always consider the category of modules in $\operatorname{Fil}(\mathcal{C})$ over a certain $\mathbb{E}_\infty$-algebra. This strategy also works for the ∞-category of *Borel G-spectra* when G is a finite group. In [AKBBK23], Angelini-Knoll, Behrens, Belmont, and Kong introduce the category of *complete artificial motivic spectra* $\widehat{\mathcal{SH}}_{G,p}^{\mathrm{art}}$ as the category of modules over a certain $\mathbb{E}_\infty$-algebra in filtered Borel G-spectra (after a suitable completion). This deformation categorifies *the modified Adams-Novikov spectral sequences* for Borel G-spectra, which are defined as the $E[G]$-based Adams spectral sequences.

On the other hand, however, instead of merely being a category of modules in the category of filtered p-complete spectra, the category of cellular $\mathbb{C}$-motivic spectra $\mathcal{S}p_{\mathbb{C}}$ admits more structure. In [GWX21], Gheorghe, Wang, and Xu define the *Chow t-structure* on the special fiber

$$\mathcal{M}od_{C\tau}((\mathcal{S}p_{\mathbb{C}})_p^\wedge)$$

and identify $\mathcal{M}od_{C\tau}((\mathcal{S}p_{\mathbb{C}})_p^\wedge)$ with the even part of the derived ∞-category of BP_*BP-comodules. In [BKWX22], Bachmann, Kong, Wang, and Xu further construct the Chow t-structure on the ∞-category of k-motivic spectra over an arbitrary base field k. Although we can interpret $(\mathcal{S}p_{\mathbb{C}})_p^\wedge$ as the p-completion of $\mathcal{M}od_{\Gamma_*(\mathbb{S})}(\operatorname{Fil}(\mathcal{S}p))$, the Chow t-structure and the identification of $C\tau$-module is hard to describe from the filtered spectra point of view.

Nevertheless, the theory of synthetic spectra provides an alternative way to understand

$$(\mathcal{S}p_{\mathbb{C}})^{\wedge}_{p} \simeq \mathcal{M}od_{\Gamma_*(\mathbb{S})^{\wedge}_{p}}(\mathcal{S}p^{\wedge}_{p})$$

topologically. In [Pst23], an E-synthetic spectrum is defined as a spherical (product-preserving) sheaf of spectra over the full subcategory of $\mathcal{S}p$ spanned by finite E-projective spectra with the E_*-epimorphism topology. In particular, the category of E-synthetic spectra admits a natural t-structure as a category of sheaves of spectra.

Moreover, Pstrągowski proves that, after p-completion, the category of cellular $\mathbb{C}$-motivic spectra is equivalent to the category of *even MU-synthetic spectra* $\mathcal{S}yn^{\mathrm{even}}_{MU}$, a full subcategory of $\mathcal{S}yn_{MU}$ generated under colimits by the bigraded spheres with even Chow degrees. In particular, the Chow t-structure on $\mathcal{S}p_{\mathbb{C}}$ coincides with the natural t-structure on $\mathcal{S}yn^{\mathrm{even}}_{MU}$. In analogy to [GWX21], Pstrągowski also shows a comparison between the category of $C\tau$-modules in $\mathcal{S}yn_E$ with the derived ∞-category of E_*E-comodules.

Along this sheaf-theoretical strategy, Patchkoria and Pstrągowski [PP23] further construct the perfect derived ∞-category of a stable ∞-category $\mathcal{C}$ together with a well-behaved homology functor $H : \mathcal{C} \to \mathcal{A}$ into an Abelian category $\mathcal{A}$. This categorifies the H-Adams spectral sequences in $\mathcal{C}$. Although the perfect derived ∞-category is not presentable in general, in [Bur22], Burklund modifies this construction and provides a presentable deformation whenever the generic fiber $\mathcal{C}$ is compactly generated, and this fits into the Definition 1.1.

However, a priori, the construction of synthetic spectra is not a special case of perfect derived ∞-categories, as pointed out in [PP23, Remark 6.58]. This paper provides a general sheaf-theoretical construction of deformations that includes the category of synthetic spectra and perfect derived ∞-categories as examples.

In summary, given a presentably symmetric monoidal stable ∞-category $\mathcal{C}$, we have introduced two main methods to construct a deformation with the generic fiber being equivalent to $\mathcal{C}$: as a category of modules in $\mathrm{Fil}(\mathcal{C})$, or as a category of spherical sheaves of spectra over a certain ∞-site.

It is natural to ask whether there is a comparison between these two methods. In fact, in [Law24], Lawson proves that the category of E-synthetic spectra $\mathcal{S}yn_E$ is equivalent to a category of modules in $\mathrm{Fil}(\mathcal{S}p)$ over a certain $\mathbb{E}_\infty$-algebra whenever E is connective. In this paper, we will show a comparison between these two constructions for the ∞-category of (genuine) G-spectra $\mathcal{S}p_G$ in Corollary 4.19. However, a comparison for a general ground category $\mathcal{C}$ is still mysterious. It relies on other structures, for example, Assumption 2.35.

1.2. **Main construction.** Following the theory of excellent ∞-sites studied in [Pst23, Section 2], we construct a machine that produces deformations for general ground categories as generic fibers.

Let $\mathcal{C} \in \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$ be a rigidly generated, presentably symmetric monoidal stable ∞-category. Let $\tilde{\mathcal{C}}$ be a full subcategory of $\mathcal{C}$ and $\mathcal{Q}$ be a collection of morphisms in $\tilde{\mathcal{C}}$.

Theorem 1.8 (2.3, 2.29, 2.31, 2.33)**.** *Suppose that the triple $(\mathcal{C}, \tilde{\mathcal{C}}, \mathcal{Q})$ satisfies the conditions in Assumptions 2.1. Then, $\mathcal{Q}$ determines a Grothendieck pretopology on $\tilde{\mathcal{C}}$, and the category of spherical sheaves of spectra*

$$\mathrm{Def}(\mathcal{C}, \mathcal{Q}) := Sh^{\mathcal{S}p}_{\Sigma}(\tilde{\mathcal{C}})$$

admits a structure of being an $\mathbb{E}_\infty$-monoidal one-parameter deformation.

Moreover, the generic fiber

$$\mathcal{S}p \otimes_{\mathrm{Fil}(\mathcal{S}p)} \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \simeq \mathcal{C} \in \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$$

is equivalent (as $\mathcal{S}p$-algebras) to $\mathcal{C}$.

The special fiber admits a well-behaved t-structure such that its left separated part

$$\mathcal{M}od^{\mathrm{sep}}_{C\tau}(\mathrm{Def}(\mathcal{C}, \mathcal{Q})) \simeq \mathcal{D}(Sh^{\mathcal{S}et}_{\Sigma}(\tilde{C})) \in \mathrm{CAlg}_{\mathrm{Gr}(\mathcal{S}p)}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$$

is equivalent (as $\mathrm{Gr}(\mathcal{S}p)$-algebras) to the (unbounded) derived ∞-category of its heart.

In order to read off the information of spectral sequences in $\mathcal{C}$ from $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$, we define the synthetic analogue functor.

Theorem 1.9 (2.11, 2.12, 2.25, 2.27)**.** *The synthetic analogue functor $\nu : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$, defined as the stabilization of the sheafification of the Yoneda embedding, satisfies the following conditions:*

(1) the composite

$$\mathcal{C} \xrightarrow{\nu} \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \xrightarrow{\tau^{-1}} \mathcal{C}$$

is the identity on $\mathcal{C}$;

(2) ν is fully faithful and preserves filtered colimits;

(3) ν is lax symmetric monoidal and is strongly symmetric monoidal after restricted to the full subcategory $\tilde{\mathcal{C}}$.

In addition, we compare the construction of $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ with the perfect derived ∞-categories and filtered objects, each of which requires more assumptions on the ground category $\mathcal{C}$.

Theorem 1.10 (3.13)**.** *Under the conventions in Assumptions 2.1, suppose that $\tilde{\mathcal{C}}$ and $\mathcal{Q}$ satisfy Assumptions 3.12; that is, $\tilde{\mathcal{C}}$ is the full subcategory $\mathcal{C}^\omega$ of $\mathcal{C}$ spanned by compact objects and $\mathcal{Q}$ is an epimorphism class in $\mathcal{C}^\omega$ with enough injectives. Then the deformation category* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ *is equivalent to the* Ind*-completion of the perfect derived ∞-category of $\mathcal{C}^\omega$ on the epimorphism class $\mathcal{Q}$.*

Theorem 1.11 (2.38, 2.39)**.** *Under the conventions in Assumptions 2.1, suppose that $\mathcal{C}$ is further equipped with a symmetric monoidal left adjoint*

$$c : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$$

that satisfies Assumption 2.35; namely,

$$\tau^{-1} \circ c = id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}.$$

Then tensoring with c provides an adjoint pair

$$\mathrm{Fil}(\mathcal{C}) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Def}(\mathcal{C}, \mathcal{Q})$$

with left adjoint i^* being a map between $\mathrm{Fil}(\mathcal{S}p)$*-algebras.*

Moreover, let the category of cellular objects $\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}}$ *be the essential image of the left adjoint i^*. Then the restriction of the above adjoint pair on* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}}$ *induces an equivalence*

$$\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}} \simeq \mathcal{M}od_{i_* \mathbb{1}_{\mathrm{Def}(\mathcal{C}, \mathcal{Q})}}(\mathrm{Fil}(\mathcal{C})).$$

Our next goal is to apply the above construction to produce a deformation with the generic fiber being equivalent to the ∞-category of (genuine) G-spectra $\mathcal{S}p_G$ for any finite group G such that it categorifies the ($\mathbb{Z}$-graded, Mackey-functor-valued) homotopy fixed point spectral sequences in $\mathcal{S}p_G$.

By Lemma 4.5, the category $\mathcal{S}p_G$, the full subcategory $\mathcal{S}p_G^\omega$ spanned by compact G-spectra, the collection $\mathcal{Q}$ of $\mathbb{S}[G]$-split morphisms in $\mathcal{S}p_G^\omega$, together with the symmetric monoidal left adjoint

$$c : \mathcal{S}p_G \to \mathcal{S}yn_{\mathbb{S}[G]}$$

constructed in Proposition 4.8, satisfy Assumptions 2.1, 3.12, and 2.35. Therefore, we define the ∞-category $\mathcal{S}yn_{\mathbb{S}[G]}$ of $\mathbb{S}[G]$-*synthetic spectra* to be the deformation category $\mathrm{Def}(\mathcal{S}p_G, \mathcal{Q}) \simeq Sh_\Sigma^{\mathcal{S}p}(\mathcal{S}p_G^\omega)$.

Now we explain the relation between $\mathcal{S}yn_{\mathbb{S}[G]}$ and the homotopy fixed point spectral sequences.

Theorem 1.12 (4.19)**.** *Consider the adjoint pair*

$$\mathrm{Fil}(\mathcal{S}p_G) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{S}yn_{\mathbb{S}[G]}$$

obtained from Theorem 1.11. For any G-spectra $X \in \mathcal{S}p_G$, let $AF_*(X) \in \mathrm{Fil}(\mathcal{S}p_G)$ be the $\mathbb{S}[G]$-Adams filtration of X defined in Definition 4.15, which coincides with the filtration for the homotopy fixed point spectral sequence. Then, we have an equivalence

$$i_*(\nu X) \simeq \mathrm{D\acute{e}c} AF_*(X) \in \mathrm{Fil}(\mathcal{S}p_G).$$

Here, $\mathrm{D\acute{e}c} : \mathrm{Fil}(\mathcal{S}p_G) \to \mathrm{Fil}(\mathcal{S}p_G)$ *is defined using the standard t-structure on $\mathcal{S}p_G$.*

In particular, let $\mathcal{S}yn_{\mathbb{S}[G]}^{\mathrm{cell}}$ be the full subcategory of $\mathcal{S}yn_{\mathbb{S}[G]}$ generated by

$$\{\Sigma^{m,n}\nu(\Sigma^\infty_+ T) : m, n \in \mathbb{Z}, T \in \mathbb{F}_G\}$$

under colimits. Then we have

$$\mathcal{S}yn_{\mathbb{S}[G]}^{\mathrm{cell}} \simeq \mathcal{M}od_{\mathrm{D\acute{e}c}AF_*(\mathbb{S})}(\mathrm{Fil}(\mathcal{S}p_G)).$$

Remark 1.13. By the lax symmetric monoidality of the right adjoint i_*, the décalage of the $\mathbb{S}[G]$-Adams filtration

$$\mathcal{S}p_G \xrightarrow{AF_*} \mathrm{Fil}(\mathcal{S}p_G) \xrightarrow{\mathrm{D\acute{e}c}} \mathrm{Fil}(\mathcal{S}p_G)$$

is canonically lax symmetric monoidal. In particular, the $\mathbb{Z}$-graded Mackey-functor-valued homotopy fixed point spectral sequence of an $\mathbb{E}_\infty$-algebra in $\mathcal{S}p_G$ admits a multiplicative structure. This provides a new proof of the classical theorem on the monoidality of the homotopy fixed point spectral sequences (e.g. [HR24]).

1.3. **Organization of the paper.** In Section 2, we introduce a sheaf-theoretical construction of deformations based on Pstrągowski's theory of excellent ∞-sites [Pst23]. In particular, we provide descriptions of the generic fiber and the special fiber. We end this section with a comparison between this sheaf-theoretical construction and filtered objects.

In Section 3, we recall the construction of perfect derived ∞-categories [PP23]. Then, we prove that over well-behaved ground categories, our sheaf-theoretical construction coincides with Burklund's modification of derived ∞-categories [Bur22].

In Section 4, we apply the sheaf-theoretical construction to the ∞-category of genuine G-spectra $\mathcal{S}p_G$ for a finite group G and introduce the category of $\mathbb{S}[G]$-synthetic spectra $\mathcal{S}yn_{\mathbb{S}[G]}$ together with the synthetic analogue functor $\nu : \mathcal{S}p_G \to \mathcal{S}yn_{\mathbb{S}[G]}$. In particular, we get an adjoint pair between the category of filtered G-spectra and $\mathcal{S}yn_{\mathbb{S}[G]}$. We finish this section by showing that along the right adjoint $\mathcal{S}yn_{\mathbb{S}[G]} \to \mathcal{S}p_G$, the synthetic analogue of a G-spectrum X is sent to the décalage of its homotopy fixed point filtration.

Appendix A provides a brief introduction on filtered objects and décalage. In particular, we prove the coincidence between the cosimplicial décalage functor defined in [Lev15] and the sequential décalage defined in [Hed20].

1.4. **Conventions and notations.** Throughout this paper, we always assume that G is a finite group.

Unless otherwise specified, by a category we always indicate an ∞-category.

Given a presentably monoidal ∞-category $\mathcal{C}$ and a $\mathcal{C}$-enriched category $\mathcal{D}$, we denote the $\mathcal{C}$-valued internal hom-object in $\mathcal{D}$ by $\mathrm{map}^{\mathcal{C}}_{\mathcal{D}}(-,-)$ or simply $\mathrm{map}^{\mathcal{C}}(-,-)$ if the ground category $\mathcal{D}$ is clear.

In particular, if $\mathcal{C}$ is one of the ∞-categories of

sets, spaces, spectra, filtered spectra, and G-spectra,

we abbreviate $\mathrm{map}^{\mathcal{C}}(-,-)$ by

$$\mathrm{Hom}(-,-),\ \mathrm{map}(-,-),\ \mathrm{F}(-,-),\ \mathrm{F}_*(-,-),\ \text{and}\ \mathrm{F_G}(-,-),$$

respectively.

Moreover, we use the following notations:

$\mathcal{P}r^{\mathrm{L}}$	the category of presentable ∞-categories and left adjoints
$\mathcal{P}r^{\mathrm{R}}$	the category of presentable ∞-categories and right adjoints
$\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}$	the full subcategory of $\mathcal{P}r^{\mathrm{L}}$ spanned by stable ∞-categories
$\mathrm{CAlg}(\mathcal{C})$	$\mathbb{E}_\infty$-algebras in a symmetric monoidal category $\mathcal{C}$
$\mathcal{M}od_R(\mathcal{C})$	R-modules in $\mathcal{C}$
$\mathbb{1}_{\mathcal{C}}$	the unit in a symmetric monoidal category $\mathcal{C}$
$P^{\mathcal{C}}(-)$	$\mathcal{C}$-valued presheaves
$P^{\mathcal{C}}_{\Sigma}(-)$	$\mathcal{C}$-valued spherical presheaves
$Sh^{\mathcal{C}}(-)$	$\mathcal{C}$-valued sheaves
$Sh^{\mathcal{C}}_{\Sigma}(-)$	$\mathcal{C}$-valued spherical sheaves

In the last four cases, we omit the superscript $\mathcal{C}$ if $\mathcal{C} \simeq \mathcal{S}$ is the category of spaces.

1.5. **Acknowledgments.** This paper is the author's graduation thesis at Peking University. I am very grateful to my advisor, Guchuan Li, for suggesting this problem and for his advice. I am also grateful to Weinan Lin for having introduced me to the theory of synthetic spectra. I would like to thank Yuchen Wu, from whom I learned the comparison between the two definitions of décalage. Finally, it is a pleasure to thank Kaiyi Chen, Yifan Jin, Hana Jia Kong, Jikun Li, Yuxuan Li, Zhenpeng Li, Wei Yang, Cloudifold Zhang, and Jinchen Zhao for the useful conversations and helpful comments.

2. A sheaf-theoretical construction of deformations

In this section, we generalize the construction of synthetic spectra in [Pst23] to obtain a sheaf-theoretical machine that produces deformations.

Our construction is slightly different, but closely related to the perfect derived ∞-categories defined in [PP23]. We will explain this in the next section.

2.1. **Spherical sheaves of spectra over excellent ∞-sites.** In [Pst23, Section 2], Pstrągowski thoroughly studies the theory of spherical sheaves of spectra over excellent ∞-sites. We will apply this theory to define the deformations for more general stable ∞-categories under Assumptions 2.1 as generic fibers. In addition, we slightly generalize some propositions stated in [Pst23].

Assumptions 2.1. Throughout Section 2, let $\mathcal{C} \in \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$ be a rigidly generated, presentably symmetric monoidal stable ∞-category. Here we say that a presentably symmetric monoidal stable ∞-category is rigidly generated if it is compactly generated in which an object is compact if and only if it is dualizable.

Let $\tilde{\mathcal{C}}$ be a full subcategory of $\mathcal{C}$ such that

(1) all elements in $\tilde{\mathcal{C}}$ are compact in $\mathcal{C}$;
(2) the zero object $0 \in \mathcal{C}$ in contained in $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ is closed under finite products, suspensions, and desuspensions;
(3) the unit $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ in contained in $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}$ is closed tensor product;
(4) $\mathcal{C}$ is generated by $\tilde{\mathcal{C}}$ under all colimits.

Moreover, let $\mathcal{Q} \subset \mathrm{Fun}(\Delta^1, \tilde{\mathcal{C}})$ be a class of morphisms in $\tilde{\mathcal{C}}$ such that

(1) all equivalences are in $\mathcal{Q}$;
(2) $\mathcal{Q}$ is closed under composites;
(3) given any $x \to y$ in $\mathcal{Q}$ and an arbitrary map $z \to y$, their pull-back (computed in $\mathcal{C}$) $x \times_y z$ is contained in $\tilde{\mathcal{C}}$ and $x \times_y z \to z$ is in $\mathcal{Q}$;
(4) $\mathcal{Q}$ is closed under tensoring with any $x \in \tilde{\mathcal{C}}$.

Definition 2.2. We define a Grothendieck pretopology on $\tilde{\mathcal{C}}$ such that each covering family consists of a single morphism and $\{f : X \to Y\}$ is a covering if and only if f is contained in $\mathcal{Q}$.

Lemma 2.3. *Under Definition 2.2, $\tilde{\mathcal{C}}$ forms an excellent ∞-site in the sense of* [Pst23, Definition 2.24]*; that is, an additive ∞-site (i.e. a small ∞-site which is additive as an ∞-category and such that every covering family consists of a single map) equipped with a symmetric monoidal structure such that all objects admit duals and such that for any $c \in \tilde{\mathcal{C}}$, the functor $- \otimes c : \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$ takes coverings to coverings.*

Proof. This follows automatically from Assumptions 2.1. ☐

Now we introduce the definition of the deformation category $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$.

Definition 2.4. Let $\mathrm{Def}(\mathcal{C}, \mathcal{Q}) := Sh^{\mathcal{S}p}_{\Sigma}(\tilde{\mathcal{C}})$ be the ∞-category of spherical (i.e. product-preserving) sheaves of spectra over $\tilde{\mathcal{C}}$.

Example 2.5. Let $E \in \mathcal{S}p$ be a ring spectrum of Adams-type. Then we set $\mathcal{C} := \mathcal{S}p$ to be the ∞-category of spectra, $\tilde{\mathcal{C}} := \mathcal{S}p^{\mathrm{fp}}_E$ to be the full subcategory of finite E-projective spectra, and $\mathcal{Q}$ to be the class of E_*-epimorphisms. In this case, $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ coincides with the ∞-category $\mathcal{S}yn_E$ of E-synthetic spectra.

The following properties of $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ hold directly from the theory on excellent ∞-sites.

Proposition 2.6. *Let $Sh_\Sigma(\tilde{\mathcal{C}})$ be the ∞-category of spherical sheaves of spaces over $\tilde{\mathcal{C}}$. Then* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ *is the stabilization of $Sh_\Sigma(\tilde{\mathcal{C}})$. Moreover, both $Sh_\Sigma(\tilde{\mathcal{C}})$ and* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ *admit presentably symmetric monoidal structures through Day convolution.*

Proof. The first statement follows from [Pst23, Proposition 2.13] and the well-definedness of the Day convolution follows from [Pst23, Proposition 2.30] and [Lura, 2.2.1.9]. □

Furthermore, the canonical t-structure on $\mathcal{S}p$ determines a point-wise t-structure on $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ following [Pst23, Definition 2.15].

Definition 2.7. If $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is a spherical sheaf of spectra over $\tilde{\mathcal{C}}$, then its *n-th homotopy group* $\pi_n X$ is a sheaf of discrete abelian groups obtained as the sheafification of the presheaf

$$P \in \tilde{\mathcal{C}} \mapsto \pi_n X(P)$$

We say that a spherical sheaf of spectra X in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is *connective* if $\pi_n X = 0$ for $n \leqslant 0$. We say that it is *coconnective* if $\Omega^\infty X$ is a discrete sheaf of spaces. We denote the subcategories of connective and coconnective spherical sheaves by $\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\geqslant 0}$ and $\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\leqslant 0}$, respectively.

Proposition 2.8. $(\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\geqslant 0}, \mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\leqslant 0})$ *defines a right complete t-structure on* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ *which is compatible with filtered colimits and the symmetric monoidal structure. Moreover, there is a canonical equivalence* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})^\heartsuit \simeq Sh_\Sigma^{\mathcal{S}et}(\tilde{\mathcal{C}})$ *between the heart of this t-structure and the category of spherical sheaves of sets.*

Proof. This is exactly [Pst23, Proposition 2.16.] □

Although not mentioned in [Pst23], this t-structure interacts well with the sheafification.

Lemma 2.9. *The sheafification functor $L : P^{\mathcal{S}p}(\tilde{\mathcal{C}}) \to Sh^{\mathcal{S}p}(\tilde{\mathcal{C}})$ is t-exact. Here, $P^{\mathcal{S}p}(\tilde{\mathcal{C}})$ is equipped with the point-wise t-structure; that is, the n-covers and n-truncations are all computed level-wise. An analogous statement holds for the spherical sheafification $L : P_\Sigma^{\mathcal{S}p}(\tilde{\mathcal{C}}) \to Sh_\Sigma^{\mathcal{S}p}(\tilde{\mathcal{C}})$.*

Proof. Note that a presheaf is exactly a sheaf on the trivial Grothendieck topology on $\tilde{\mathcal{C}}$. Therefore, the non-spherical statement holds from [Lurb, Remark 1.3.2.8] since the sheafification of sheaves of spaces $L : P(\tilde{\mathcal{C}}) \to Sh(\tilde{\mathcal{C}})$ is a geometric morphism between ∞-topoi. Also, since n-covers, n-truncations, and sheafification preserve the property of being spherical, we deduce the spherical statement. □

In fact, we can further identify the connective part of $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ with the category of sheaves of spaces on $\tilde{\mathcal{C}}$. Therefore, $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is precisely the stabilization of the prestable ∞-category $Sh_\Sigma(\tilde{\mathcal{C}})$.

Proposition 2.10. *Consider the adjunction $\Sigma_+^\infty \dashv \Omega^\infty : Sh_\Sigma(\tilde{\mathcal{C}}) \rightleftarrows \mathrm{Def}(\mathcal{C}, \mathcal{Q})$, where Ω^∞ is computed level-wise. Then, the left adjoint Σ_+^∞ is fully faithful, symmetric monoidal, and identifies its domain with the ∞-category* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\geqslant 0}$ *of connective spherical sheaves of spectra.*

Proof. Note that the monoidality of Σ_+^∞ holds by the construction of the Day convolution. See [Pst23, Proposition 2.30]. Then this statement follows from [Pst23, Proposition 2.19]. □

Given the identification of $\mathrm{Def}(\mathcal{C},\mathcal{Q})_{\geqslant 0}$ with $Sh_\Sigma(\tilde{\mathcal{C}})$, we define the synthetic analogue functor.

Definition 2.11. We define the *synthetic analogue* functor $\nu : \mathcal{C} \to \mathrm{Def}(\mathcal{C},\mathcal{Q})$ to be the following composite:

$$\mathcal{C} \xrightarrow{y} P_\Sigma(\tilde{\mathcal{C}}) \xrightarrow{L} Sh_\Sigma(\tilde{\mathcal{C}}) \xrightarrow{\Sigma^\infty_+} \mathrm{Def}(\mathcal{C},\mathcal{Q}),$$

where y is the Yoneda embedding and L is the sheafification.

Proposition 2.12. *The synthetic analogue functor ν is lax symmetric monoidal and preserves filtered colimits.*

Proof. By definition, ν is the composition

$$\mathcal{C} \xrightarrow{y} P_\Sigma(\tilde{\mathcal{C}}) \xrightarrow{L} Sh_\Sigma(\tilde{\mathcal{C}}) \xrightarrow{\Sigma^\infty_+} \mathrm{Def}(\mathcal{C},\mathcal{Q}).$$

The last two functors are symmetric monoidal left adjoints and thus preserve filtered-colimits. The Yoneda embedding preserves filtered colimits because objects in $\tilde{\mathcal{C}}$ are compact in $\mathcal{C}$. This shows that the composition ν is filtered-colimit preserving.

To see that the Yoneda embedding y is lax symmetric monoidal, observe that it has a left adjoint $P_\Sigma(\tilde{\mathcal{C}}) \to \mathcal{C}$ which is the unique colimit-preserving functor that extends the inclusion $\tilde{\mathcal{C}} \to \mathcal{C}$. The inclusion is symmetric monoidal, so the left adjoint is symmetric monoidal by the universal property of Day convolution. Then y itself is lax symmetric monoidal. □

In [Pst23, Theorem 2.8], Pstrągowski states and proves a recognition theorem of spherical sheaves. It would be more convenient to state a stable version of this theorem to recognize spherical sheaves of spectra. Indeed, Pstrągowski's original proof also works in the stable case.

Theorem 2.13. *A spherical presheaf of spectra $X \in P_\Sigma^{\mathcal{S}p}(\tilde{\mathcal{C}})$ is a sheaf if and only if it satisfies the following exactness property: if $F \to B \to A$ is a fiber sequence in $\tilde{\mathcal{C}}$ with $B \to A$ in $\mathcal{Q}$, then $X(A) \to X(B) \to X(F)$ is a fiber sequence of spectra.*

Proof. Recall that a presheaf is a sheaf if and only if it sends the Čech nerve of every covering family to a limit diagram. See [Pst23, Corollary A.9].

First note that for any $B \to A$ in $\mathcal{Q}$, its fiber F in $\mathcal{C}$ is contained in $\tilde{\mathcal{C}}$ by Assumptions 2.1.

Let X be a spherical presheaf that satisfies the above exactness property and let $B \to A$ be any map in $\mathcal{Q}$. We need to check

$$X(A) \to X(B) \rightrightarrows X(B \times_A B) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

is a limit diagram.

We first reduce this to the special cases when $B = A$ or $A = 0$. To do this, consider the following map between fiber sequences:

$$\begin{array}{ccccc} F & \longrightarrow & B & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & A \end{array}$$

In this diagram, all vertical maps are in $\mathcal{Q}$ since $F \to 0$ is the pull-back of $B \to A$ along $0 \to A$. Taking the Čech nerve of these vertical maps and applying X, we get

$$\begin{tikzcd} X(0) \arrow[r] & X(F) \arrow[r, shift left] \arrow[r, shift right] & X(F \times_0 F) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \\ X(A) \arrow[r] \arrow[u] & X(B) \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & X(B \times_A B) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[u] & \cdots \\ X(A) \arrow[r] \arrow[u] & X(A) \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & X(A \times_A A) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[u] & \cdots \end{tikzcd}$$

Here, all columns are induced by applying X to a fiber sequence, since the fiber commutes with pull-backs. Also, since $\mathcal{Q}$ is closed along pull-backs and $B \to A$ is in $\mathcal{Q}$, we get all

$$B \times_A B \times_A \cdots \times_A B \to A \times_A A \times_A \cdots \times_A A = A$$

are in $\mathcal{Q}$. Therefore, by the exactness property, all vertical sequences in the above diagram are fiber sequences.

Since totalization also commutes with fiber, we get the following map between fiber sequences

$$\begin{tikzcd} X(A) \arrow[r] \arrow[d] & X(A) \arrow[r] \arrow[d] & X(0) \arrow[d] \\ \operatorname{Tot} X(A \times_A \cdots \times_A A) \arrow[r] & \operatorname{Tot} X(B \times_A \cdots \times_A B) \arrow[r] & \operatorname{Tot} X(F \times_0 \cdots \times_0 F) \end{tikzcd}$$

Therefore, to show that the middle vertical map is an equivalence, it suffices to show that the left and right vertical maps are.

The left one is obviously an equivalence, since in that cosimplicial object, all objects are $X(A)$ and all higher morphisms are identity, and thus the totalization has to be $X(A)$.

To show that the right one is an equivalence, since X is spherical and is valued in $\mathcal{S}p$, we have $X(0) = 0$ and $X(F \times_0 \cdots \times_0 F) = X(F) \oplus \cdots \oplus X(F)$. Then it suffices to show that

$$0 = \operatorname{Tot} X(F)^{\oplus n},$$

which is equivalent to showing that $0 \to X(F)$ is an effective monomorphism. This holds automatically as all morphisms in a stable ∞-category are effective monomorphisms.

Conversely, suppose X is a spherical sheaf and let $B \to A$ be a map in $\mathcal{Q}$ with fiber F. Consider the following map of fiber sequences

$$\begin{tikzcd} F \arrow[r] \arrow[d] & B \times_A B \arrow[r] \arrow[d] & B \arrow[d] \\ F \arrow[r] & B \arrow[r] & A \end{tikzcd}$$

Here, all vertical maps are also in $\mathcal{Q}$ since $\mathcal{Q}$ is closed under pull-back.

Now we also take the Čech nerves of all vertical maps. Then we get

$$\begin{array}{ccccccc}
X(F) \to & X(F) & \rightrightarrows & X(F \times_F F) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots \\
\uparrow & \uparrow & & \uparrow & \\
X(B) \to & X(B \times_A B) & \rightrightarrows & X((B \times_A B) \times_B (B \times_A B)) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots \\
\uparrow & \uparrow & & \uparrow & \\
X(A) \to & X(B) & \rightrightarrows & X(B \times_A B) & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots
\end{array}$$

Here, all horizontal augmented cosimplicial objects are limit diagrams since X satisfies the sheaf condition. Therefore, since totalization commutes with fiber, to show that the first column is a fiber sequence, it suffices to show that all other columns are.

In fact, all sequences

$$F \times_F \cdots \times_F F \to (B \times_A B) \times_B \cdots \times_B (B \times_A B) \to B \times_A \cdots \times_A B$$

are fiber sequences since the fiber commutes with pull-back. Moreover, all these fiber sequences are split because we can choose the splitting to be

$$\Delta \times \cdots \times \Delta : B \times_A \cdots \times_A B \to (B \times_A B) \times_B \cdots \times_B (B \times_A B)$$

where $\Delta : B \to B \times_A B$ is the diagonal. Since X is spherical, it takes split fiber sequences to split fiber sequences. This proves the whole theorem. □

Corollary 2.14. *The full subcategory* $\mathrm{Def}(\mathcal{C}, \mathcal{Q}) \hookrightarrow P^{\mathcal{S}p}(\tilde{\mathcal{C}})$ *is closed under colimits. Therefore, all colimits in* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ *are computed level-wise.*

Proof. Since $\mathcal{S}p$ is stable, all colimits commute with finite products, so the full subcategory of spherical presheaves of spectra $P^{\mathcal{S}p}_{\Sigma}(\tilde{\mathcal{C}})$ is closed under colimits. Moreover, by Theorem 2.13, requiring a spherical presheaf of spectra to be a sheaf is equivalent to requiring a collection of sequences in $\mathcal{S}p$ to be fiber sequences. All colimits also commute with fiber sequences in the stable ∞-category $\mathcal{S}p$, so the corollary follows. □

2.2. **The deformation parameter τ.** To verify that $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is a deformation in the sense of Definition 1.1, we first introduce a bi-grading on $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ and define the deformation parameter $\tau : \Sigma^{0,-1} \to id$.

Definition 2.15. For any $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ and integers $a, b \in \mathbb{Z}$, we define $\Sigma^{a,b}X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ to be the following sheaf: for any $P \in \tilde{\mathcal{C}}$, we set

$$(\Sigma^{a,b}X)(P) := \Sigma^{-b}X(\Sigma^{-a-b}P).$$

This indeed satisfies the sheaf conditions by the recognition theorem Theorem 2.13.

Remark 2.16. This bi-grading convention coincides with the Adams grading in the Adams spectral sequences. In our convention, the categorical suspension functor is $\Sigma^{1,-1}$ and the synthetic analogue interacts with the grading as $\nu\Sigma = \Sigma^{1,0}\nu$. The latter statement will be shown in the proof of Proposition 2.19.

Definition 2.17. For any $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$, we define the degree $(0, -1)$ self-map $\tau_X : \Sigma^{0,-1}X \to X$ as follows.

For any $P \in \tilde{\mathcal{C}}$, applying X to the canonical push-out diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma P \end{tikzcd}$$

we get

$$\begin{tikzcd} X(P) & 0 \arrow[l] \\ 0 \arrow[u] & X(\Sigma P) \arrow[l] \arrow[u] \end{tikzcd}$$

which further induces a canonical map $(\Sigma^{0,-1}X)(P) = \Sigma X(\Sigma P) \to X(P)$. This canonical map is functorial on P and we denote the resulting map of sheaves $\Sigma^{0,-1}X \to X$ by τ_X.

The above constructions are actually smashing on X. That is, if we denote $\mathbb{1}^{a,b} := \Sigma^{a,b}\nu\mathbb{1}_{\mathcal{C}}$ and abbreviate $\tau_{\mathbb{1}^{0,0}} : \mathbb{1}^{0,-1} \to \mathbb{1}^{0,0}$ by τ, then we have $\Sigma^{a,b}X = \mathbb{1}^{a,b} \otimes X$ and $\tau_X = \tau \otimes X$ for all $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$. To see this, we need to first recall a lemma from [Pst23].

Lemma 2.18. *Let c be any object in $\tilde{\mathcal{C}}$ with its dual c^*. Then the Day convolution functor $\nu(c) \otimes - : \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is naturally equivalent to the precomposition functor along $- \otimes c^* : \tilde{\mathcal{C}} \to \tilde{\mathcal{C}}$.*

Proof. First, note that the precomposition functor along $- \otimes c^*$ sends a spherical sheaf to a spherical sheaf by Theorem 2.13. By [Pst23, Lemma 2.27], these two functors coincide on the connective part $Sh_\Sigma(\tilde{\mathcal{C}}) \simeq \mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\geqslant 0}$. To be more precise, as self-functors on $P(\tilde{\mathcal{C}})$, we have

$$y(c) \otimes y(d)(e) = \mathrm{map}(e, c \otimes d) = \mathrm{map}(e \otimes c^*, d) = y(d)(e \otimes c^*)$$

for all $d, e \in \tilde{\mathcal{C}}$. Both $y(c) \otimes -$ and the precomposition functor along $- \otimes c^*$ preserve colimits, since colimits in $P(\tilde{\mathcal{C}})$ are computed level-wisely. Therefore, the two functors coincide on $P(\tilde{\mathcal{C}})$ as $P(\tilde{\mathcal{C}})$ is generated by the representatives $y(d)$ under colimits.

By [Pst23, Theorem 2.8], the unstable version of Theorem 2.13, we know that the precomposition functor along $- \otimes c^*$ sends spherical sheaves to spherical sheaves. Thus, the two functors also coincide on $Sh_\Sigma(\tilde{\mathcal{C}})$ by viewing $Sh_\Sigma(\tilde{\mathcal{C}})$ as a full subcategory of $P(\tilde{\mathcal{C}})$. Then, identifying $Sh_\Sigma(\tilde{\mathcal{C}})$ with $\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\geqslant 0}$ along Σ^∞_+, we get this lemma on the connective part.

To get the general statement, first note that the two self-functors on $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ commute with the categorical suspension. So, indeed, we have already proved this lemma on the bounded below part. For a general $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$, since the t-structure on $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is right complete, we can always write X as a sequential colimit of n-covers of X

$$X = \operatorname*{colim}_n \tau_{\geqslant n} X.$$

Therefore, it suffices to show that both of the two functors commute with sequential colimits. This holds for $y(c) \otimes -$ since $- \otimes -$ preserves colimits on each variable. This is also true for the precomposition functor along $- \otimes c^*$, since colimits in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ are computed level-wise by Corollary 2.14. $\square$

Now we can state and prove the desired property.

Proposition 2.19. *Let* $\mathbb{1}^{a,b} := \Sigma^{a,b}\nu\mathbb{1}_{\mathcal{C}}$ *and* $\tau := \tau_{\mathbb{1}^{0,0}} : \mathbb{1}^{0,-1} \to \mathbb{1}^{0,0}$. *Then, we have* $\Sigma^{a,b}X = \mathbb{1}^{a,b} \otimes X$ *and* $\tau_X = \tau \otimes X : \mathbb{1}^{0,-1} \otimes X \to \mathbb{1}^{0,0} \otimes X$ *for all* $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$.

Proof. By Lemma 2.18, we have

$$(\Sigma^{a,b}X)(P) = \Sigma^{-b}X(\Sigma^{-a-b}P) = (\Sigma^{-b}\nu(\Sigma^{a+b}\mathbb{1}_{\mathcal{C}}) \otimes X)(P).$$

Setting $X = \nu\mathbb{1}_{\mathcal{C}}$ in the above equation, we get $\mathbb{1}^{a,b} = \Sigma^{-b}\nu(\Sigma^{a+b}\mathbb{1}_{\mathcal{C}})$, and thus we have $\Sigma^{a,b}X = \mathbb{1}^{a,b} \otimes X$. In particular, this implies that $\nu\Sigma = \Sigma^{1,0}\nu$ as this is smashing and holds for the unit.

To see the second statement, note that by definition τ_X is the canonical colimit-to-limit map induced by applying X to the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma P = \Sigma\mathbb{1}_{\mathcal{C}} \otimes P \end{tikzcd}$$

By Lemma 2.18, we can identify precomposing along $\Sigma\mathbb{1}_{\mathcal{C}} \otimes -$ with tensoring with $\nu(\Sigma^{-1}\mathbb{1}_{\mathcal{C}})$. Also, by Corollary 2.14 we have colimits in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ are computed level-wise. Thus, we get τ_X is the canonical colimit-to-limit map induced by the following diagram

$$\begin{tikzcd} \nu(\Sigma^{-1}\mathbb{1}_{\mathcal{C}}) \otimes X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

Then the desired proposition follows as the above diagram comes from applying $- \otimes X$ to

$$\begin{tikzcd} \nu(\Sigma^{-1}\mathbb{1}_{\mathcal{C}}) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \nu\mathbb{1}_{\mathcal{C}} \end{tikzcd}$$

which induces $\tau : \mathbb{1}^{0,-1} \to \mathbb{1}^{0,0}$ by setting $X = \nu\mathbb{1}_{\mathcal{C}}$ in the above diagram. □

Given this proposition, one can check that the bi-grading interacts well with the t-structure. Indeed, we have the following result.

Lemma 2.20. *If* $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ *is* n*-connective, then* $\Sigma^{a,b}X$ *is* $(a-b)$*-connective. An analogous result holds for coconnectivity.*

Proof. By definition of the t-structure, $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is n-coconnective if and only if X is n-coconnective level-wise. So by

$$(\Sigma^{a,b}X)(P) := \Sigma^{-b}X(\Sigma^{-a-b}P),$$

we have the self-equivalence $\Sigma^{a,b}$ induces an equivalence between $\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\leqslant n}$ and $\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\leqslant n-b}$. Hence, $\Sigma^{a,b}$ commutes with the localization functor onto the two subcategories; that is,

$$\Sigma^{a,b}\tau_{\leqslant n}X = \tau_{\leqslant n-b}\Sigma^{a,b}X.$$

Since $\Sigma^{a,b}$ is exact, we also have

$$\Sigma^{a,b}\tau_{\geqslant n}X = \tau_{\geqslant n-b}\Sigma^{a,b}X.$$

This ends the proof. ☐

2.3. **τ-invertible objects.** In this section, we show that $\mathrm{Def}(\mathcal{C},\mathcal{Q})$ defines a deformation with the generic fiber equivalent to $\mathcal{C}$, as desired. Following Remark 1.7, we first study the localization on

$$\{\tau_X : \Sigma^{0,-1}X \to X : X \in \mathrm{Def}(\mathcal{C},\mathcal{Q})\}.$$

Definition 2.21. An object $X \in \mathrm{Def}(\mathcal{C},\mathcal{Q})$ is called *τ-invertible* if $\tau_X : \Sigma^{0,-1}X \to X$ is an equivalence. We denote the full subcategory of τ-invertible objects by $\mathrm{Def}(\mathcal{C},\mathcal{Q})(\tau^{-1})$.

Remark 2.22. Indeed, an object $Y \in \mathrm{Def}(\mathcal{C},\mathcal{Q})$ is τ-invertible if and only if it is local on the collection of morphisms

$$\{\tau_X : \Sigma^{0,-1}X \to X : X \in \mathrm{Def}(\mathcal{C},\mathcal{Q})\}.$$

To see this, recall that τ is self-dual up to a degree shift. Then, Y is local on

$$\{\tau_X : \Sigma^{0,-1}X \to X : X \in \mathrm{Def}(\mathcal{C},\mathcal{Q})\}$$

if and only if

$$\begin{aligned} &\mathrm{map}(X,Y) \xrightarrow[\simeq]{\tau_X} \mathrm{map}(\Sigma^{0,-1}X,Y) \text{ for all } X \in \mathrm{Def}(\mathcal{C},\mathcal{Q}) \\ \iff &\mathrm{map}(X,\Sigma^{0,-1}Y) \xrightarrow[\simeq]{\tau_Y} \mathrm{map}(X,Y) \text{ for all } X \in \mathrm{Def}(\mathcal{C},\mathcal{Q}) \\ \iff &\tau_Y : \Sigma^{0,-1}Y \simeq Y. \end{aligned}$$

Example 2.23. The first examples of τ-invertible objects are those belonging to the image of the spectral Yoneda embedding Y. Let $X \in \tilde{\mathcal{C}}$ be any object and consider the presheaf of spectra represented by X:

$$Y(X) : P \mapsto \mathrm{F}(P,X).$$

This presheaf $Y(X)$ is a sheaf by Theorem 2.13 since $\mathrm{F}(-,X)$ preserves finite products and fiber sequences as $\mathcal{C}$ is stable. In addition, $Y(X)$ is always τ-invertible since $\mathrm{F}(-,X)$ preserves push-out diagrams and thus sends

$$\begin{tikzcd} P \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma P \end{tikzcd}$$

to a push-out diagram.

We will later show that the spectral Yoneda embedding $Y : \mathcal{C} \to \mathrm{Def}(\mathcal{C},\mathcal{Q})$ actually provides all τ-invertible objects.

Before that, we first consider some basic properties of $\mathrm{Def}(\mathcal{C},\mathcal{Q})(\tau^{-1})$.

Proposition 2.24. *The inclusion* $\mathrm{Def}(\mathcal{C},\mathcal{Q})(\tau^{-1}) \hookrightarrow \mathrm{Def}(\mathcal{C},\mathcal{Q})$ *admits a left adjoint* $\tau^{-1} : \mathrm{Def}(\mathcal{C},\mathcal{Q}) \to \mathrm{Def}(\mathcal{C},\mathcal{Q})(\tau^{-1})$. *Moreover,* $\mathrm{Def}(\mathcal{C},\mathcal{Q})(\tau^{-1})$ *admits a unique symmetric monoidal structure that preserves colimits on each variable such that the left adjoint τ^{-1} is symmetric monoidal. Therefore, $\tau^{-1}\nu\mathbb{1}_{\mathcal{C}}$ admits an $\mathbb{E}_\infty$-algebra structure in* $\mathrm{Def}(\mathcal{C},\mathcal{Q})$ *and we have a symmetric monoidal equivalence*

$$\mathrm{Def}(\mathcal{C},\mathcal{Q})(\tau^{-1}) \simeq \mathcal{M}od_{\tau^{-1}\nu\mathbb{1}_{\mathcal{C}}}(\mathrm{Def}(\mathcal{C},\mathcal{Q})).$$

Proof. We define $\tau^{-1} : \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ as

$$\tau^{-1}X := \operatorname{colim}(X \xrightarrow{\tau} \Sigma^{0,1}X \xrightarrow{\tau} \Sigma^{0,2}X \xrightarrow{\tau} \cdots).$$

Then $\tau^{-1}X$ is automatically τ-invertible, as this sequential colimit is computed level-wise.

To check that τ^{-1} is the desired left adjoint, let $Y \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ be any τ-invertible object. Then, we have

$$\operatorname{map}(\tau^{-1}X, Y) \simeq \lim \operatorname{map}(\Sigma^{0,n}X, Y) \simeq \lim \operatorname{map}(X, \Sigma^{0,-n}Y) \simeq \operatorname{map}(X, Y)$$

Here, the last equivalence holds since Y is τ-invertible and the middle equivalence holds from the fact that the dual of $\tau : \mathbb{1}^{0,-1} \to \mathbb{1}^{0,0}$ is exactly $\Sigma^{0,1}\tau : \mathbb{1}^{0,0} \to \mathbb{1}^{0,1}$. To see this, recall that ν is symmetric monoidal when restricted to $\tilde{\mathcal{C}}$ and thus preserves duals. Moreover, in the proof of Proposition 2.19 we have shown that $\tau : \mathbb{1}^{0,-1} \to \mathbb{1}^{0,0}$ is the canonical colimit-to-limit comparison map induced by applying ν to the push-out diagram

$$\begin{tikzcd} \Sigma^{-1}\mathbb{1} \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathbb{1} \end{tikzcd}$$

Taking the dual of this diagram, we get the desired equivalence.

So far, we have constructed a localization functor

$$\tau^{-1} : \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1}).$$

To induce a symmetric monoidal structure on $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$, by [Lura, Proposition 2.2.1.9], it suffices to check $-\otimes X$ preserves τ^{-1}-equivalences for any $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$. As τ^{-1} being a left adjoint, the collection of τ^{-1}-equivalences is closed under colimits. Furthermore, since the connective part of $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ can be identified with $Sh_\Sigma(\tilde{\mathcal{C}})$, $\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\geqslant 0}$ is generated by $\{\nu(P) : P \in \tilde{\mathcal{C}}\}$ under colimits. Therefore, $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is generated by desuspensions of $\{\nu(P) : P \in \tilde{\mathcal{C}}\}$ under colimits since the t-structure on $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is right complete. Hence, it suffices to check $- \otimes \nu(P)$ preserves τ^{-1}-equivalences for any $P \in \tilde{\mathcal{C}}$.

Taking the cofiber of the input τ^{-1}-equivalence, this is equivalent to saying that $\tau^{-1}(X \otimes \nu(P)) \simeq 0$ whenever $\tau^{-1}(X) \simeq 0$. Note that by Lemma 2.18, $- \otimes \nu(P)$ can be identified with precomposing with the dual of P, so the statement holds as the colimit

$$\tau^{-1}X := \operatorname{colim}(X \xrightarrow{\tau} \Sigma^{0,1}X \xrightarrow{\tau} \Sigma^{0,2}X \xrightarrow{\tau} \cdots)$$

is computed level-wise.

Therefore, the symmetric monoidality of τ^{-1} implies that its right adjoint, the inclusion $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is lax symmetric monoidal, and thus sends the unit of $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ to an $\mathbb{E}_\infty$-algebra in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$, which is precisely $\tau^{-1}\mathbb{1}^{0,0}$.

To identify $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ with $\mathcal{M}od_{\tau^{-1}\nu\mathbb{1}_{\mathcal{C}}}(\mathrm{Def}(\mathcal{C}, \mathcal{Q}))$, by [BHS22, Proposition A.4], it suffices to check:

(1) $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is rigidly generated; or equivalently, $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ admits a collection of compact dualizable generators;
(2) The inclusion $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ preserves colimits;
(3) The inclusion $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is conservative.

Here, statement (3) holds automatically as the inclusion is fully faithful. To see statement (1), note that $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is generated by desuspensions of $\{\nu(P) : P \in \tilde{\mathcal{C}}\}$ and $\nu(P)$ is dualizable as ν is symmetric monoidal when restricted to $\tilde{\mathcal{C}}$. Also, to see $\nu(P)$ is compact, by adjunction, for any filtered diagram X_α in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$, we have

$$\mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})}(\nu(P), \operatorname*{colim}_\alpha X_\alpha) \simeq \mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})_{\geqslant 0}}(\nu(P), \operatorname*{colim}_\alpha \tau_{\geqslant 0} X_\alpha)$$

Here, we are implicitly using the fact that the t-structrue on $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is compatible with filtered colimits. See Proposition 2.8. Therefore, statement (1) holds because $\nu(P)$ is compact in $Sh_\Sigma(\tilde{\mathcal{C}})$ as filtered colimits in $Sh_\Sigma(\tilde{\mathcal{C}})$ are computed level-wise. See [Pst23, Corollary 2.9].

Lastly, to show statement (2), it is equivalent to showing that the full subcategory $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ of $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is closed under colimits. This also holds from the fact that colimits in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ are computed level-wise. Indeed, the colimit of a diagram of equivalences $\tau : \Sigma X_\alpha(\Sigma P) \to X_\alpha(P)$ is also an equivalence. This ends the proof. □

Recall that we have defined $\nu : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ as the composition of the sheafification of the Yoneda embedding y together with the stabilization functor $\Sigma^\infty_+ : Sh_\Sigma(\tilde{\mathcal{C}}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$. In practice, it is more convenient to reinterpret this synthetic analogue functor in several different ways.

Proposition 2.25. *Let $\nu : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ be the synthetic analogue. Then the following statements hold:*

(1) *For any $X \in \tilde{\mathcal{C}}$, νX is the sheafification of the presheaf*

$$P \mapsto \mathrm{F}(P, X)_{\geqslant 0}.$$

Hence, there is a canonical natural transformation $\nu X \to Y(X)$.

(2) *The canonical map $\nu X \to Y(X)$ exhibits νX as the 0-connective cover $Y(X)_{\geqslant 0}$. Moreover, we have $Y(X)_{\geqslant n} \simeq \Sigma^{0,-n}\nu X$ for all integer n and the Whitehead tower of $Y(X)$ can be identified with*

$$\cdots \xrightarrow{\tau} \Sigma^{0,n}\nu X \xrightarrow{\tau} \Sigma^{0,n+1}\nu X \xrightarrow{\tau} \cdots \to Y(X).$$

In particular, $Y(X) \simeq \tau^{-1}\nu X$.

(3) *$C\tau^n \simeq (\nu\mathbb{1}_\mathcal{C})_{\leqslant n-1}$ admits a unique $\mathbb{E}_\infty$-algebra structure in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ that is compatible with the canonical map $\nu\mathbb{1}_\mathcal{C} \to C\tau^n$.*

Proof. (1) By [Lurb, C.1.5.3, C.1.5.8], a spherical presheaf over X is canonically a spherical presheaf of infinite loop spaces, and thus a spherical presheaf of connective spectra.

Therefore, it suffices to show that the diagram

$$\begin{tikzcd} P_\Sigma(\tilde{\mathcal{C}}) \arrow[r, "\Sigma^\infty_+"] \arrow[d] & P_\Sigma^{\mathcal{S}p}(\tilde{\mathcal{C}}) \arrow[d] \\ Sh_\Sigma(\tilde{\mathcal{C}}) \arrow[r, "\Sigma^\infty_+"] & Sh_\Sigma^{\mathcal{S}p}(\tilde{\mathcal{C}}) \end{tikzcd}$$

commutes. Here, both vertical functors are sheafifications. Note that a presheaf is a sheaf with respect to the trivial topology, so by [Pst23, Proposition 2.19], the top horizontal functor is computed by level-wise delooping an infinite loop space to a connective spectrum, and the bottom horizontal functor is computed by level-wise delooping an infinite loop space to a connective spectrum and then sheafify.

However, taking the right adjoints of all functors in the above diagram, we get

$$\begin{tikzcd} P_\Sigma(\tilde{\mathcal{C}}) & P_\Sigma^{\mathcal{S}p}(\tilde{\mathcal{C}}) \arrow[l, "\Omega^\infty"'] \\ Sh_\Sigma(\tilde{\mathcal{C}}) \arrow[u] & Sh_\Sigma^{\mathcal{S}p}(\tilde{\mathcal{C}}) \arrow[l, "\Omega^\infty"'] \arrow[u] \end{tikzcd}$$

which automatically commutes as Ω^∞ is always computed level-wise. Then the diagram of left adjoints also commutes by the equivalence $\mathcal{P}r^{\mathrm{L}} \simeq (\mathcal{P}r^{\mathrm{R}})^{op}$ from [Lur09, Corollary 5.5.3.4]. Therefore, the statement follows since $\mathrm{map}(P, X) \simeq \Omega^\infty \mathrm{F}(P, X)_{\geqslant 0}$.

(2) This follows immediately from (1) as the sheafification functor is t-exact by Lemma 2.9 since an analogous statement holds in $P_\Sigma^{\mathcal{S}p}(\tilde{\mathcal{C}})$.

(3) The statement holds since $C\tau^n$ is the $(n-1)$-truncation $(\nu\mathbb{1}_{\mathcal{C}})_{\leqslant n-1}$ of the unit $\nu\mathbb{1}_{\mathcal{C}}$ by (2). $\square$

Now we can state and prove the main theorem of this section.

Theorem 2.26. *The spectral Yoneda embedding* $Y : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ *is fully faithful with essential image* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$*. Moreover, the equivalence*

$$Y : \mathcal{C} \simeq \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1}) \simeq \mathcal{M}od_{\tau^{-1}\nu\mathbb{1}_{\mathcal{C}}}(\mathrm{Def}(\mathcal{C}, \mathcal{Q}))$$

is canonically symmetric monoidal.

Proof. By Corollary 2.14, all colimits and limits in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ are computed level-wise since the inclusion $\mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to P^{\mathcal{S}p}(\tilde{\mathcal{C}})$ is a right adjoint.

By definition,

$$Y(X) : P \to \mathrm{F}(P, X)$$

preserves all limits and thus preserves finite colimits since $\mathcal{C}$ is stable. Moreover, Y also preserves filtered colimits as $P \in \tilde{\mathcal{C}}$ are compact. Therefore, $Y : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ preserves all limits and colimits.

As shown in the proof of Proposition 2.24, the subcategory $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is closed under colimits and also under limits since the inclusion is a right adjoint. Therefore, the restriction $Y : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ also preserves all limits and colimits.

Firstly, we show that $Y : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ is fully faithful; that is,

$$\mathrm{map}_{\mathcal{C}}(A, B) \to \mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})(\tau^{-1})}(Y(A), Y(B))$$

is an equivalence for all $A, B \in \mathcal{C}$. If $A \simeq P$ for some $P \in \tilde{\mathcal{C}}$, then we have

$$\begin{aligned} \mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})(\tau^{-1})}(Y(P), Y(B)) &\simeq \mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})}(\nu(P), Y(B)) \\ &\simeq \mathrm{map}_{Sh_\Sigma(\tilde{\mathcal{C}})}(y(P), \Omega^\infty Y(B)) \\ &\simeq \Omega^\infty \mathrm{F}_{\mathcal{C}}(P, B) \\ &\simeq \mathrm{map}_{\mathcal{C}}(P, B). \end{aligned}$$

For general $A \in \mathcal{C}$, since Y preserves colimits, the collection of all $A \in \mathcal{C}$ with $\mathrm{map}_{\mathcal{S}p_G}(A, B) \simeq \mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})(\tau^{-1})}(Y(A), Y(B))$ for a fixed $B \in \mathcal{C}$ is closed under colimits. Thus, $Y : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ is fully faithful since $\tilde{\mathcal{C}}$ generates $\mathcal{C}$ under colimits by Assumptions 2.1.

To show that $Y : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ is an equivalence, first note that both $\mathcal{C}$ and $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ are presentable since the latter is the module category in a presentably symmetric monoidal category $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$. Then, the colimit-preserving functor Y admits a right adjoint $R : \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$. As Y is fully faithful, it suffices to show that for any $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$, $RX \simeq 0$ implies $X \simeq 0$, and hence the cofiber of the counit $YRX \to X$ is always trivial.

To see this, assume $RX \simeq 0$ for some τ-invertible object X. Then

$$0 = \mathrm{map}(P, RX) \simeq \mathrm{map}(Y(P), X) \simeq \mathrm{map}(\nu P, X) \simeq \Omega^\infty X(P)$$

for all $P \in \tilde{\mathcal{C}}$, so $X(P)$ is (-1)-truncated for all P. As X τ-invertible, we have

$$X(P) \simeq \Sigma^{-n} X(\Sigma^{-n} P)$$

with the right-hand side $(-n-1)$ truncated. Thus, $X(P)$ is $(-n-1)$-truncated for all n so it has to be zero.

It remains to show the symmetric monoidality of Y. By identifying $\mathcal{C}$ with $\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ along Y, the τ-inversion $\tau^{-1} : \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})$ can be identified with the left adjoint $L : \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathcal{C}$ of $Y : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$. Then, it suffices to show that $L : \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathcal{C}$ admits a symmetric monoidal structure.

Since $\tau^{-1}\nu X = Y(X)$, we have $L\nu X = X$. Hence, L is the unique colimit-preserving exact extension of $\tilde{\mathcal{C}} \to \mathcal{C}$ along $\nu : \tilde{\mathcal{C}} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$. By the universal property of Day convolution, the symmetric monoidal inclusion $\tilde{\mathcal{C}} \to \mathcal{C}$ induces a canonical symmetric monoidal structure on $L : \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathcal{C}$. This ends the proof. $\square$

As a corollary, we have the following.

Corollary 2.27. *The synthetic analogue $\nu : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is fully faithful.*

Proof. Since the spectral Yoneda embedding Y is fully faithful, it suffices to show that

$$\tau^{-1} : \mathrm{map}(\nu(A), \nu(B)) \to \mathrm{map}(Y(A), Y(B))$$

is an equivalence for any $A, B \in \mathcal{C}$.

Indeed,

$$\begin{aligned} &\mathrm{map}_{\mathrm{Def}(\mathcal{C}, \mathcal{Q})}(\nu(A), \nu(B)) \\ \simeq\ &\mathrm{map}_{\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\geqslant 0}}(\nu(A), \nu(B)) \\ \simeq\ &\mathrm{map}_{\mathrm{Def}(\mathcal{C}, \mathcal{Q})}(\nu(A), Y(B)) \\ \simeq\ &\mathrm{map}_{\mathrm{Def}(\mathcal{C}, \mathcal{Q})(\tau^{-1})}(Y(A), Y(B)) \\ \simeq\ &\mathrm{map}_{\mathrm{Def}(\mathcal{C}, \mathcal{Q})}(Y(A), Y(B)). \end{aligned}$$

This ends the proof. $\square$

Remark 2.28. We now show that the τ-inversion functor $\tau^{-1} : \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathcal{C}$ exhibits $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ as a deformation with the generic fiber equivalent to $\mathcal{C}$ in the sense of Definition 1.1. To see this, we first introduce a $\mathrm{Fil}(\mathcal{S}p)$-algebra structure on $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$, which is equivalently a symmetric monoidal functor

$$i : \mathbb{Z}^{\leqslant} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q}).$$

Following [BHS22, Construction C.18], we construct the functor

$$i : \mathbb{Z}^{\leqslant} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$$

such that i sends $n \in \mathbb{Z}^{\leqslant}$ to $\mathbb{1}^{0,n}$ and sends morphisms in $\mathbb{Z}^{\leqslant}$ to (compositions of) τ.

To see i is symmetric monoidal, let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ spanned by the unit $\mathbb{1}_{\mathcal{C}}$ and $\mathcal{D}_{\mathrm{def}}$ be the full subcategory of $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ spanned by the objects $\mathbb{1}^{0,n}$. Since $\mathcal{D}_{\mathrm{def}}$ and $\mathcal{D}$ are closed under the tensor product, they are both symmetric monoidal categories.

Let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}_{\mathrm{def}} \times \mathbb{Z}^{\leqslant}$ spanned by objects $(\mathbb{1}^{0,n}, n)$. Then $\mathcal{D}'$ is also closed under the tensor product. We claim that the composition

$$\mathcal{D}' \to \mathcal{D}_{\mathrm{def}} \times \mathbb{Z}^{\leqslant} \xrightarrow{\tau^{-1} \times \mathbb{Z}^{\leqslant}} \mathcal{D} \times \mathbb{Z}^{\leqslant}$$

is an equivalence.

This is automatically a bijection on objects. On mapping spaces, we have

$$\mathrm{map}_{\mathcal{D}'}((\mathbb{1}^{0,n}, n), (\mathbb{1}^{0,m}, m)) \simeq \begin{cases} \mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})}(\mathbb{1}^{0,n}, \mathbb{1}^{0,m}) & \text{if } n \leqslant m; \\ \varnothing & \text{if } n > m, \end{cases}$$

and

$$\mathrm{map}_{\mathcal{D} \times \mathbb{Z}^{\leqslant}}((\mathbb{1}, n), (\mathbb{1}, m)) \simeq \begin{cases} \mathrm{map}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}) & \text{if } n \leqslant m; \\ \varnothing & \text{if } n > m. \end{cases}$$

When $n \leqslant m$, we have

$$\begin{aligned} \mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})}(\mathbb{1}^{0,n}, \mathbb{1}^{0,m}) &\simeq \mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})}(Y(\mathbb{1}_{\mathcal{C}})_{\geqslant -n}, Y(\mathbb{1}_{\mathcal{C}})_{\geqslant -m}) \\ &\simeq \mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})}(Y(\mathbb{1}_{\mathcal{C}})_{\geqslant -n}, Y(\mathbb{1}_{\mathcal{C}})) \\ &\simeq \mathrm{map}_{\mathrm{Def}(\mathcal{C},\mathcal{Q})}(Y(\mathbb{1}_{\mathcal{C}}), Y(\mathbb{1}_{\mathcal{C}})) \\ &\simeq \mathrm{map}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}) \end{aligned}$$

and the equivalence is given by the τ-inversion. Hence, we have a symmetric monoidal equivalence $\mathcal{D}' \to \mathcal{D} \times \mathbb{Z}^{\leqslant}$.

The desired functor i is the composition of symmetric monoidal functors

$$i : \mathbb{Z}^{\leqslant} \longrightarrow \mathcal{D} \times \mathbb{Z}^{\leqslant} \xleftarrow{\simeq} \mathcal{D}' \hookrightarrow \mathcal{D}_{\mathrm{def}} \times \mathbb{Z}^{\leqslant} \longrightarrow \mathcal{D}_{\mathrm{def}} \hookrightarrow \mathrm{Def}(\mathcal{C}, \mathcal{Q}).$$

Extending $i : \mathbb{Z}^{\leqslant} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ to $P(\mathbb{Z}^{\leqslant})$ and then stabilizing, we get a symmetric monoidal left adjoint

$$\mathrm{Fil}(\mathcal{S}p) \simeq P^{\mathcal{S}p}(\mathbb{Z}^{\leqslant}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q}).$$

This exhibits $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ as a $\mathrm{Fil}(\mathcal{S}p)$-algebra in $\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}$.

Moreover, since i sends morphisms in $\mathbb{Z}^{\leqslant}$ to the compositions of τ, it follows that $\tau : \Sigma^{0,-1} \to id$ coincides with the deformation parameter defined in Remark 1.7.

By Remark 1.7 and Proposition 2.24, the induced functor

$$\mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathcal{S}p \otimes_{\mathrm{Fil}(\mathcal{S}p)} \mathrm{Def}(\mathcal{C}, \mathcal{Q})$$

is precisely the localization on $\tau_X : \Sigma^{0,-1}X \to X$, which coincides with the τ-inversion functor $\tau^{-1} : \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathcal{C}$.

Therefore, we have shown the following result.

Theorem 2.29. *Under the conventions in Assumptions 2.1,* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ *is a* $\mathbb{E}_\infty$*-monoidal one-parameter deformation and its generic fiber*

$$\mathcal{S}p \otimes_{\mathrm{Fil}(\mathcal{S}p)} \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \simeq \mathcal{C} \in \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$$

is equivalent to $\mathcal{C}$ *as presentably symmetric monoidal stable* ∞*-categories.*

2.4. **The special fiber.** In the previous section, we have shown that the deformation category $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is a deformation with the generic fiber being

$$\mathcal{S}p \otimes_{\mathrm{Fil}(\mathcal{S}p)} \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \simeq \mathcal{C}.$$

In this section, we provide a description of the special fiber $\mathcal{M}od_{C\tau}(\mathrm{Def}(\mathcal{C}, \mathcal{Q}))$.

Remark 2.30. We are abusing the notation $C\tau$ with different meanings: as the $\mathbb{E}_\infty$-algebra in $\mathrm{Fil}(\mathcal{S}p)$, or the $\mathbb{E}_\infty$-algebra in $(\nu\mathbb{1}_{\mathcal{C}})_{\leqslant 0}$ in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$. Here, the $\mathbb{E}_\infty$-algebra structure on $(\nu\mathbb{1}_{\mathcal{C}})_{\leqslant 0}$ is induced by the 0-truncation.

Since $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is an $\mathrm{Fil}(\mathcal{S}p)$-algebra, we have an structure map

$$i : \mathrm{Fil}(\mathcal{S}p) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q}).$$

By definition, the underlying object of $i(C\tau)$ in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is $(\nu\mathbb{1}_{\mathcal{C}})_{\leqslant 0}$. Furthermore, they also coincide as $\mathbb{E}_\infty$-algebras in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ by Proposition 2.25. Therefore, the special fiber $\mathcal{M}od_{C\tau}(\mathrm{Def}(\mathcal{C}, \mathcal{Q}))$ is precisely the category of modules over $(\nu\mathbb{1}_{\mathcal{C}})_{\leqslant 0} \in \mathrm{CAlg}(\mathrm{Def}(\mathcal{C}, \mathcal{Q}))$ in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$.

To begin with, we introduce the t-structure on the special fiber.

Proposition 2.31. *The t-structure on* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ *determines a right complete t-structure on* $\mathcal{M}od_{C\tau}(\mathrm{Def}(\mathcal{C}, \mathcal{Q}))$ *that is compatible with filtered colimits and the symmetric monoidal structure. More precisely, a $C\tau$-module is connective, or coconnective, if and only if its underlying object in* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ *is.*

In particular, the heart of $\mathcal{M}od_{C\tau}(\mathrm{Def}(\mathcal{C}, \mathcal{Q}))$ *is equivalent to* $Sh_\Sigma^{\mathcal{S}et}(\tilde{\mathcal{C}})$.

Proof. If we replace $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ by $Sh^{\mathcal{S}p}(\tilde{C})$, then the statement is a corollary of [Lurb, Proposition 2.1.1.1]. In fact, the same proof holds for spherical sheaves of spectra $\mathrm{Def}(\mathcal{C}, \mathcal{Q}) \simeq Sh_\Sigma^{\mathcal{S}p}(\tilde{C})$.

To compute the heart, we have

$$\mathcal{M}od_{C\tau}^{\heartsuit}(\mathrm{Def}(\mathcal{C}, \mathcal{Q})) \simeq \mathcal{M}od_{\pi_0 C\tau}(\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\heartsuit}) \simeq \mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\heartsuit} \simeq Sh_\Sigma^{\mathcal{S}et}(\tilde{\mathcal{C}}).$$

This ends the proof. ☐

Notations 2.32. For any $\mathcal{D} \in \mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}$ that admits a t-structure which is compatible with filtered colimits, the full subcategory spanned by ∞-connective objects $\mathcal{D}_{\geqslant\infty}$ is a localizing subcategory since it is closed under colimits and desuspensions. Then, we define the left separated part $\mathcal{D}^{\mathrm{sep}}$ of $\mathcal{D}$ as the full subcategory spanned by $\mathcal{D}_{\geqslant\infty}$-local objects.

Now we state and proof the main theorem of this section.

Theorem 2.33. *The left separated part of the special fiber*

$$\mathcal{M}od_{C\tau}^{\mathrm{sep}}(\mathrm{Def}(\mathcal{C}, \mathcal{Q})) \simeq \mathcal{D}(Sh_\Sigma^{\mathcal{S}et}(\tilde{C}))$$

is equivalent to the (unbounded) derived ∞-category of the heart $Sh_\Sigma^{\mathcal{S}et}(\tilde{C})$.

Moreover, this equivalence is symmetric monoidal, and we can further equip the derived ∞-category with a $\mathrm{Gr}(\mathcal{S}p)$*-algebra structure through the above equivalence.*

Proof. Firstly, note that the heart $Sh_\Sigma^{\mathcal{S}et}(\tilde{C})$ is Grothendieck abelian since the t-structure on $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is compatible with filtered colimits.

Then, according to [Lurb, Remark C.5.4.11], the unbounded derived ∞-category $\mathcal{D}(Sh_\Sigma^{\mathcal{S}et}(\tilde{C}))$ together with its t-structure is determined by the following properties:

(1) $\mathcal{D}(Sh_\Sigma^{\mathcal{S}et}(\tilde{C}))$ is a presentable stable ∞-category;

(2) The t-structure on $\mathcal{D}(Sh_{\Sigma}^{\mathcal{S}et}(\tilde{C}))$ is right complete, left separated, and compatible with filtered colimits;
(3) The Grothendieck prestable ∞-category $\mathcal{D}(Sh_{\Sigma}^{\mathcal{S}et}(\tilde{C}))_{\geqslant 0}$ is 0-complicial;
(4) $\mathcal{D}(Sh_{\Sigma}^{\mathcal{S}et}(\tilde{C}))^{\heartsuit} \simeq Sh_{\Sigma}^{\mathcal{S}et}(\tilde{C})$.

By Proposition 2.31, it suffices to show that the connective part of the left separated part of $\mathcal{M}od_{C\tau}(\mathrm{Def}(\mathcal{C}, \mathcal{Q}))$ is 0-complicial. According to [Lurb, Proposition C.5.3.3], the property of being 0-complicial is preserved under taking left exact localizations. Therefore, it suffices to show that

$$\mathcal{M}od_{C\tau}(\mathrm{Def}(\mathcal{C}, \mathcal{Q}))_{\geqslant 0} \simeq \mathcal{M}od_{C\tau}(\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\geqslant 0})$$

is 0-complicial; that is, it is generated by 0-truncated objects under colimits.

Since $\mathcal{M}od_{C\tau}(\mathrm{Def}(\mathcal{C}, \mathcal{Q})_{\geqslant 0}) \simeq \mathcal{M}od_{C\tau}(Sh_{\Sigma}(\tilde{\mathcal{C}}))$, we know that it is generated under colimits by the representatives $C\tau \otimes \nu(P)$ for $P \in \tilde{\mathcal{C}}$, and $C\tau \otimes \nu(P)$ are indeed 0-truncated by Proposition 2.25.

To show the symmetric monoidality, by the right completeness and since the t-structure is compatible with the symmetric monoidal structure, it suffices to prove the prestable case; that is, to show the symmetric monoidality of the equivalence

$$\mathcal{M}od_{C\tau}^{\mathrm{sep}}(\mathrm{Def}(\mathcal{C}, \mathcal{Q}))_{\geqslant 0} \simeq \mathcal{D}(Sh_{\Sigma}^{\mathcal{S}et}(\tilde{C}))_{\geqslant 0}$$

is symmetric monoidal.

The left-hand side is generated under colimits by (the hypercompletions of)

$$\mathcal{B} := \{C\tau \otimes \nu(P) : P \in \tilde{\mathcal{C}}\},$$

which is closed under tensor products because ν is symmetric monoidal when restricted to $\tilde{\mathcal{C}}$.

Along the above equivalence, this collection $\mathcal{B}$ is corresponded to

$$\mathcal{B}' := \{y(P) \in Sh_{\Sigma}^{\mathcal{S}et}(\tilde{\mathcal{C}}) \subset \mathcal{D}(Sh_{\Sigma}^{\mathcal{S}et}(\tilde{C}))_{\geqslant 0} : P \in \tilde{\mathcal{C}}\}$$

where y is the Yoneda embedding. We claim that the collection $\mathcal{B}'$ is also closed under tensor products.

To see this, by the universal property of Day convolution, the Yoneda embedding $y : \tilde{\mathcal{C}} \to Sh_{\Sigma}^{\mathcal{S}et}(\tilde{\mathcal{C}})$ is symmetric monoidal. Since the objects in $\tilde{\mathcal{C}}$ are all dualizable, their images along y are also dualizable. Note that tensoring with a dualizable object preserves small limits, the objects in $\mathcal{B}'$ are all flat as objects in the Abelian category $Sh_{\Sigma}^{\mathcal{S}et}(\tilde{\mathcal{C}})$. Therefore, the derived tensor product of two objects in $\mathcal{B}'$ remains discrete and thus contains in $\mathcal{B}'$.

Now, it suffices to show that the equivalence between the full subcategories spanned by $\mathcal{B}$ and $\mathcal{B}'$ is symmetric monoidal. This holds because both categories are 1-categories and we have the symmetric monoidality of $\nu|_{\tilde{\mathcal{C}}}$ and y. This ends the proof. $\square$

Remark 2.34. It is natural to ask for a description of the entire special fiber, instead of the left separated part. For example, one may expect that the special fiber is equivalent to the unseparated derived ∞-category. For the category of E-synthetic spectra, this holds if E is Landweber exact or the sphere spectrum. See [Pst23, Proposition 4.53]. However, we even do not know whether this is true for all Adams-type homotopy ring spectra E.

2.5. **Comparison with filtered objects.** In this section, we provide a comparison between the category $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ and the category of filtered objects in $\mathcal{C}$. In [BHS22, Appendix C], Burklund, Hahn, and Senger constructed a machine to deform homotopy theories from the filtered objects viewpoint. We will mainly follow their strategy.

More precisely, we would like to compare filtered objects in $\mathcal{C}$ and the category $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ through a morphism between $\mathrm{Fil}(\mathcal{S}p)$-algebras

$$\mathrm{Fil}(\mathcal{C}) \simeq \mathrm{Fil}(\mathcal{S}p) \otimes \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q});$$

that is, a $\mathrm{Fil}(\mathcal{S}p)$-linear symmetric monoidal left adjoint. As $\mathrm{Fil}(\mathcal{C})$ being the free $\mathrm{Fil}(\mathcal{S}p)$-algebra of $\mathcal{C}$, such a functor is equivalently a symmetric monoidal left adjoint

$$c : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q}).$$

Moreover, we also expect the $\mathrm{Fil}(\mathcal{S}p)$-algebra map to be compatible with the identification of the generic fibers with $\mathcal{C}$; that is, we hope the following diagram to commute:

$$\begin{tikzcd} \mathrm{Fil}(\mathcal{C}) \arrow[rr, "c"] \arrow[dr, "\mathrm{colim}"'] & & \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \arrow[dl, "\tau^{-1}"] \\ & \mathcal{C} & \end{tikzcd}$$

This leads to the following assumption.

Assumption 2.35. Under Assumptions 2.1, we further assume that there exists a symmetric monoidal left adjoint $c : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ such that

$$\tau^{-1} \circ c = id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}.$$

Remark 2.36. The functor $c : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is different from ν in general since ν does not commute with suspensions.

Under Assumption 2.35, the deformation category $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ is enriched in $\mathcal{C}$.

Definition 2.37. For any $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$, we define the $\mathcal{C}$-internal hom-object

$$\mathrm{map}^{\mathcal{C}}(X, -) : \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \to \mathcal{C}$$

as the right adjoint to

$$c(-) \otimes X : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q}).$$

Here, the right adjoint exists because $\mathcal{C}$ and $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ are presentable and $c(-)\otimes X$ preserves all colimits.

Now we apply the constructions in [BHS22, Appendix C] to the deformation $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$.

Construction 2.38. By Remark 2.28, we have a symmetric monoidal functor

$$\mathrm{Fil}(\mathcal{S}p) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q}).$$

Moreover, note that the tensor product is the coproduct in $\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$ since $\mathcal{S}p$ is idempotent in $\mathcal{P}r^{\mathrm{L}}$. So, by tensoring with $c : \mathcal{C} \to \mathrm{Def}(\mathcal{C}, \mathcal{Q})$, we get a symmetric monoidal left adjoint

$$i^* : \mathrm{Fil}(\mathcal{S}p) \otimes \mathcal{C} \simeq \mathrm{Fil}(\mathcal{C}) \to \mathrm{Def}(\mathcal{C}, \mathcal{Q}).$$

Let i_* be the right adjoint to i^*, $\tilde{c}$ be the trivial filtration functor

$$\mathcal{C} \to \mathrm{Fil}(\mathcal{S}p) \otimes \mathcal{C} \simeq \mathrm{Fil}(\mathcal{C})$$

and $\Sigma^{0,-1} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Fil}(\mathcal{C})$ be the shift functor with $(\Sigma^{0,-1}X)_n = X_{n+1}$ which is also $- \otimes \Sigma^{0,-1}\tilde{c}(\mathbb{1}_{\mathcal{C}})$. Let the categorical suspension on $\mathrm{Fil}(\mathcal{C})$ be $\Sigma^{1,-1}$. Then because i^* is $\mathrm{Fil}(\mathcal{S}p)$-linear, we have the following commutative diagram of symmetric monoidal left adjoints

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\tilde{c}"] \arrow[d, equal] & \mathrm{Fil}(\mathcal{C}) \arrow[r, "\mathrm{colim}"] \arrow[d, "i^*"] & \mathcal{C} \arrow[d, equal] \\ \mathcal{C} \arrow[r, "c"] & \mathrm{Def}(\mathcal{C}, \mathcal{Q}) \arrow[r, "\tau^{-1}"] & \mathcal{C} \end{tikzcd}$$

and we also have i^* commutes with $\Sigma^{m,n}$.

Therefore, for any $X \in \mathrm{Def}(\mathcal{C}, \mathcal{Q})$, the n-th stage of i_*X is

$$(i_*X)_n \simeq \mathrm{map}^{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, (i_*X)_n) \simeq \mathrm{map}^{\mathcal{C}}(\tilde{c}\mathbb{1}_{\mathcal{C}}, \Sigma^{0,-n}i_*X) \simeq \mathrm{map}^{\mathcal{C}}(\nu\mathbb{1}_{\mathcal{C}}, \Sigma^{0,-n}X).$$

Thus,

$$i_*X \simeq \mathrm{map}^{\mathcal{C}}(\nu\mathbb{1}_{\mathcal{C}}, \Sigma^{0,-*}X)$$

and the structure maps are induced by τ.

Now we describe this comparison more precisely.

Theorem 2.39. *Consider the adjoint pair*

$$\mathrm{Fil}(\mathcal{C}) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Def}(\mathcal{C}, \mathcal{Q})$$

constructed in Construction 2.38. Let the category of cellular objects $\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}}$ *be the essential image of the left adjoint* i^**. Then the restriction of the above adjoint pair on* $\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}}$ *induces an equivalence*

$$\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}} \simeq \mathcal{M}od_{i_*\nu\mathbb{1}_{\mathcal{C}}}(\mathrm{Fil}(\mathcal{C})).$$

Proof. Consider the restricted adjoint pair

$$\mathrm{Fil}(\mathcal{C}) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}}.$$

The left adjoint i^* is symmetric monoidal, so its right adjoint i_* is lax symmetric monoidal and thus factors through $\mathcal{M}od_{i_*\nu\mathbb{1}_{\mathcal{C}}}(\mathrm{Fil}(\mathcal{C}))$.

To show that the induced map is an equivalence, by [BHS22, Proposition A.4], it suffices to check

(1) $\mathrm{Fil}(\mathcal{C})$ is rigidly generated;
(2) $\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}}$ has compact unit;
(3) the image of i^* generates $\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}}$ under colimits.

(1) holds from [Ram24, Example 4.37] since $\mathcal{C}$ is rigidly generated. (2) holds because $\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}}$ is closed under colimits in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$ and the unit $\mathbb{1}^{0,0}$ is compact in $\mathrm{Def}(\mathcal{C}, \mathcal{Q})$, which has been proved in Proposition 2.24. (3) holds by definition of $\mathrm{Def}(\mathcal{C}, \mathcal{Q})^{\mathrm{cell}}$. This ends the proof. $\square$

Remark 2.40. By [Law24], the ∞-category of synthetic spectra $\mathcal{S}yn_E$ is cellular whenever E is connective. Let $c : \mathcal{S}p \to \mathcal{S}yn_E$ in Assumption 2.35 be the unit of $\mathcal{S}yn_E \in \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$. Then Theorem 2.39 shows that $\mathcal{S}yn_E$ is equivalent to a category of modules in $\mathrm{Fil}(\mathcal{S}p)$ over some $\mathbb{E}_\infty$-algebra whenever E is connective.

3. Comparison with derived ∞-categories

In this section, we explain the relation of the deformation constructed in the last section with the notion of perfect derived ∞-categories defined in [PP23].

3.1. **Recollection on perfect derived ∞-categories.**

In [PP23], Patchkoria and Pstrągowski introduce the notion of a perfect derived ∞-category associated to an adapted homology functor. Here, an adapted homology functor consists of the following datum.

Definition 3.1. Let $\mathcal{C}$ be a stable ∞-category and $\mathcal{A}$ be an Abelian category. A functor $H : \mathcal{C} \to \mathcal{A}$ is called an *adapted homology functor* if:

(1) H is a *homology functor*; that is, H is additive and sends cofiber sequences in $\mathcal{C}$ to exact sequences in $\mathcal{A}$.
(2) $\mathcal{A}$ has enough injectives.
(3) any injective $i \in \mathcal{A}$ lifts to an associated injective $i_{\mathcal{C}}$ in the sense that $i_{\mathcal{C}}$ repensentes the functor
$$\mathrm{Hom}_{\mathcal{A}}(H(-), i) : \mathcal{C}^{op} \to \mathcal{A}b$$
in the homotopy category $h\mathcal{C}$.
(4) the structure map $H(i_{\mathcal{C}}) \to i$ (given by the image of the identity along the equivalence $[i_{\mathcal{C}}, i_{\mathcal{C}}] \cong \mathrm{Hom}_{\mathcal{A}}(H(i_{\mathcal{C}}), i)$) is an isomorphism.

Indeed, an adapted homology functor $H : \mathcal{C} \to \mathcal{A}$ depends only on the collection of maps $\{c \to d\}$ such that $H(c) \to H(d)$ is an epimorphism in $\mathcal{A}$. To be more precise, we introduce the following definitions from [PP23].

Definition 3.2 ([PP23, Definition 3.1])**.** Let $\mathcal{C}$ be a stable ∞-category. We say a class of arrows $\mathcal{E} \subset \mathrm{Fun}(\Delta^1, \mathcal{C})$ is an *epimorphism class* if:

(1) all equivalences are in $\mathcal{E}$,
(2) for any pair f, g of composable arrows, if $f, g \in \mathcal{E}$ then also $g \circ f \in \mathcal{E}$ and if $g \circ f \in \mathcal{E}$ then also $g \in \mathcal{E}$,
(3) $\mathcal{E}$ is closed under pull-backs along arbitrary maps in $\mathcal{C}$,
(4) an arrow $f : c \to d$ belongs to $\mathcal{E}$ if and only if $\Sigma f : \Sigma c \to \Sigma d$ does.

Regarding an epimorphism class as a suitable generalization of epimorphisms in an Abelian category, we can generalize the notion of being injective to the following.

Definition 3.3 ([PP23, Definition 3.8, 3.12, 3.13])**.** Let $\mathcal{E}$ be an epimorphism class on a stable ∞-category $\mathcal{C}$. We will say an arrow $c \to d$ is $\mathcal{E}$*-monic* if the canonical map $d \to \mathrm{cofib}(c \to d)$ is in $\mathcal{E}$. We say an object $i \in \mathcal{C}$ is $\mathcal{E}$*-injective* if it has the right lifting property to $\mathcal{E}$-monics; that is, if for any $\mathcal{E}$-monic $c \to d$, the induced map
$$[d, i] \to [c, i]$$
is an epimorphism of abelian groups.

Moreover, we say $\mathcal{C}$ *has enough* $\mathcal{E}$*-injectives* if for every $c \in \mathcal{C}$ there exists an $\mathcal{E}$-monic map $c \to i$ into an $\mathcal{E}$-injective.

Theorem 3.4 ([PP23, Theorem 3.24])**.** *The following two sets of data associated to an idempotent-complete stable ∞-category $\mathcal{C}$ are equivalent:*

(1) an epimorphism class $\mathcal{E}$ on $\mathcal{C}$ such that $\mathcal{C}$ has enough $\mathcal{E}$-injectives,
(2) an adapted homology theory $H\colon \mathcal{C} \to \mathcal{A}$.

Now we can state the definition of the perfect derived ∞-category $\mathcal{D}^\omega(\mathcal{C})$.

Definition 3.5. Given an adapted homology functor $H : \mathcal{C} \to \mathcal{A}$. Consider the Grothendieck pretopology on $\mathcal{C}$ defined by setting each covering family to be a collection of a single morphism $\{c \to d\}$ such that $H(c) \to H(d)$ is an epimorphism in $\mathcal{A}$. Then we define *the perfect derived ∞-category* $\mathcal{D}^\omega(\mathcal{C})$ to be the smallest subcategory of spherical sheaves of spectra on $\mathcal{C}$ containing all stabilizations of sheafifications of representables and closed under finite colimits and desuspension.

Remark 3.6. In analogy to Definition 2.7, one can define a natural t-structure on $\mathcal{D}^\omega(\mathcal{C})$ such that the connective part $\mathcal{D}^\omega_{\geqslant 0}(\mathcal{C})$ can be identified with the smallest subcategory of spherical sheaves of spaces on $\mathcal{C}$ containing all sheafifications of representables and closed under finite colimits. We call the connective part $\mathcal{D}^\omega_{\geqslant 0}(\mathcal{C})$ the *prestable perfect derived ∞-category* of $\mathcal{C}$.

Remark 3.7. Now we briefly summarize the general properties of the perfect derived ∞-category $\mathcal{D}^\omega(\mathcal{C})$. In analogy with Definitions 2.15 and 2.17, one can define a bi-grading on $\mathcal{D}^\omega(\mathcal{C})$ as

$$(\Sigma^{a,b})X(-) := \Sigma^{-b}X(\Sigma^{-a-b}-)$$

and also

$$\tau_X : \Sigma^{0,-1}X \to X.$$

Moreover, we have the following commutative diagram

$$\begin{tikzcd} & \mathcal{D}^\omega(\mathcal{C}) \arrow[dr, "\tau^{-1}"] & \\ \mathcal{C} \arrow[ur, "\nu"] \arrow[rr, equal] & & \mathcal{C} \end{tikzcd}$$

Here, ν is the stabilization of sheafification of Yoneda embedding and τ^{-1} is the localization on $\tau_X : \Sigma^{0,-1}X \to X$. This means that τ^{-1} satisfies the universal property given by inverting τ, but τ^{-1} may not admit a fully faithful right adjoint since $\mathcal{D}^\omega(\mathcal{C})$ is not necessarily presentable.

Besides, we can read off the H-Adams spectral sequences from the above diagram. Indeed, the functor

$$\mathcal{C} \xrightarrow{\nu} \mathcal{D}^\omega(\mathcal{C}) \xrightarrow{\Sigma^{0,-*}} \mathrm{Fil}(\mathcal{D}^\omega(\mathcal{C})) \xrightarrow{\mathrm{F}(\nu\mathbb{S},-)} \mathrm{Fil}(\mathcal{S}p)$$

coincides with the décalage of the H-Adams filtration of X.

Moreover, the prestable perfect derived ∞-category $\mathcal{D}^\omega_{\geqslant 0}(\mathcal{C})$ satisfies the following universal property.

Theorem 3.8 ([PP23, Theorem 5.35])**.** *Let $H : \mathcal{C} \to \mathcal{A}$ be an adapted homology theory, $\mathcal{D}^\omega_{\geqslant 0}(\mathcal{C})$ be the corresponding prestable perfect derived ∞-category, and $\mathcal{D}$ be a prestable ∞-category with finite limits. Then, the left Kan extension along $\nu\colon \mathcal{C} \to \mathcal{D}^\omega_{\geqslant 0}(\mathcal{C})$ induces an equivalence between the two following collections of data:*

(1) exact functors $G\colon \mathcal{D}^\omega_{\geqslant 0}(\mathcal{C}) \to \mathcal{D}$ of prestable ∞-categories and
(2) exact functors $G_0\colon \mathcal{A} \to \mathcal{D}^\heartsuit$ of abelian categories together with a prestable enhancement of $G_0 \circ H$; that is, an additive left exact functor $\mathcal{H} : \mathcal{C} \to \mathcal{D}$ together with a chosen equivalence $\pi_0 \circ \mathcal{H} \simeq G_0 \circ H : \mathcal{C} \to \mathcal{D}^\heartsuit$.

In most cases, the ground category $\mathcal{C}$ admits a monoidal structure, and we expect the perfect derived ∞-category $\mathcal{D}^\omega(\mathcal{C})$ to be compatible with the monoidal structure.

Definition 3.9 ([Bur22, Definition 4.2])**.** We say that an adapted homology theory $H : \mathcal{C} \to \mathcal{A}$ on a stably $\mathbb{E}_m$-monoidal category $\mathcal{C}$ (with $m \geqslant 2$) is $\otimes$-compatible if for every H-epimorphism $X \to Y$ and $Z \in \mathcal{C}$, the map $X \otimes Z \to Y \otimes Z$ is also an H-epimorphism.

Proposition 3.10 ([Bur22, Proposition 4.3])**.** *Let $H : \mathcal{C} \to \mathcal{A}$ be an $\otimes$-compatible adapted homology theory on a stably $\mathbb{E}_m$-monoidal category $\mathcal{C}$ with $m \geqslant 2$. Then $\mathcal{D}^\omega(\mathcal{C})$ admits an exact $\mathbb{E}_m$-monoidal structure, compatible with the prestable structure, such that ν and τ^{-1} are $\mathbb{E}_m$-monoidal.*

As we do not require the ground category $\mathcal{C}$ to be small, to avoid using a large universe, it is necessary to consider the subcategory of sheaves generated only by representatives under finite colimits. However, by placing more restrictions on the adapted homology functor $H : \mathcal{C} \to \mathcal{A}$, we can modify the above construction to get a presentable deformation. We will discuss two main approaches to do this in the following sections.

3.2. **Presentable deformations as** Ind**-completions.** In [Bur22, Section 4], Burklund constructs a presentable deformation when the ground category $\mathcal{C}$ is a compactly generated presentably $\mathbb{E}_m$-monoidal, stable ∞-category with $m \geqslant 2$ such that $\mathcal{C}^\omega$ is closed under tensor products. More precisely, he defines the presentable deformation as the Ind-completion of the perfect derived ∞-category over $\mathcal{C}^\omega$.

Construction 3.11 ([Bur22, Construction 4.6])**.** Let $\mathcal{C}$ be a compactly generated presentably $\mathbb{E}_m$-monoidal stable ∞-category with $m \geqslant 2$ such that $\mathcal{C}^\omega$ is closed under tensor products. As $\mathcal{C}$ is compactly generated and presentable, $\mathcal{C}^\omega$ is small and we have $\mathcal{C} = \mathrm{Ind}(\mathcal{C}^\omega)$. Moreover, since $\mathcal{C}^\omega$ is also closed under retract, we have $\mathcal{C}^\omega$ is idempotent-complete, and therefore an adapted homology theory on $\mathcal{C}^\omega$ is equivalently an epimorphism class on $\mathcal{C}^\omega$ with enough injectives by Theorem 3.4.

Now let $\mathcal{E}$ be an epimorphism class on $\mathcal{C}^\omega$ with enough $\mathcal{E}$-injectives such that the corresponding adapted homology theory is $\otimes$-compatible. Equivalently, that is, for all $Z \in \mathcal{C}^\omega$, $- \otimes Z$ preserves maps in $\mathcal{E}$.

Let $\mathcal{D}^\omega(\mathcal{C}^\omega)$ be the associated perfect derived ∞-category on $\mathcal{C}^\omega$. Then we get the following commutative diagram of $\mathbb{E}_m$-monoidal ∞-categories and $\mathbb{E}_m$-monoidal functors, as desired.

$$\begin{tikzcd} & \mathcal{D}^\omega(\mathcal{C}^\omega) \arrow[dr, "\tau^{-1}"] & \\ \mathcal{C}^\omega \arrow[ur, "\nu"] \arrow[rr, equal] & & \mathcal{C}^\omega \end{tikzcd}$$

We define the presentable deformation on $\mathcal{C}$ to be the Ind-completion of $\mathcal{D}^\omega(\mathcal{C}^\omega)$. Then by applying Ind-completion to the whole diagram, we get a commutative diagram of presentably $\mathbb{E}_m$-monoidal stable ∞-categories and $\mathbb{E}_m$-monoidal, filtered-colimit preserving functors.

$$\begin{tikzcd} & \mathrm{Ind}(\mathcal{D}^\omega(\mathcal{C}^\omega)) \arrow[dr, "\tau^{-1}"] & \\ \mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^\omega) \arrow[ur, "\nu"] \arrow[rr, equal] & & \mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}^\omega) \end{tikzcd}$$

We claim that if we let $\tilde{\mathcal{C}}$ in Assumptions 2.1 be the full subcategory of all compact objects $\mathcal{C}^\omega$ and further require $\mathcal{Q}$ to be an epimorphism class with enough injectives over $\mathcal{C}^\omega$, the deformation category $\mathrm{Def}(\mathcal{C},\mathcal{Q})$ coincides with the above construction $\mathrm{Ind}(\mathcal{D}^\omega(\mathcal{C}^\omega))$.

Assumptions 3.12. Under Assumptions 2.1, let $\tilde{\mathcal{C}}$ be the full subcategory of all compact objects $\mathcal{C}^\omega$. Note that because $\mathcal{C}$ is rigidly generated, $\mathcal{C}^\omega$ contains $\mathbb{1}_{\mathcal{C}}$ and is closed under tensor products, so $\mathcal{C}^\omega$ satisfies all requirements of $\tilde{\mathcal{C}}$. Furthermore, we require $\mathcal{Q}$ to be an epimorphism class with enough injectives on $\mathcal{C}^\omega$.

Theorem 3.13. *Under Assumptions 3.12, the resulting diagram*

$$\begin{tikzcd} & \mathrm{Ind}(\mathcal{D}^\omega(\mathcal{C}^\omega)) \arrow[dr, "\overline{\tau}^{-1}"] & \\ \mathcal{C} \arrow[ur, "\overline{\nu}"] \arrow[rr, equal] & & \mathcal{C} \end{tikzcd}$$

is equivalent to

$$\begin{tikzcd} & \mathrm{Def}(\mathcal{C},\mathcal{Q}) \arrow[dr, "\tau^{-1}"] & \\ \mathcal{C} \arrow[ur, "\nu"] \arrow[rr, equal] & & \mathcal{C} \end{tikzcd}$$

Proof. By Definition 3.5, $\mathcal{D}^\omega(\mathcal{C}^\omega)$ is the full subcategory of $\mathrm{Def}(\mathcal{C},\mathcal{Q}) \simeq Sh_\Sigma^{\mathcal{S}p}(\mathcal{C}^\omega)$ generated by $\{\nu(P) : P \in \mathcal{C}^\omega\}$ under desuspensions and finite colimits. As $\mathrm{Def}(\mathcal{C},\mathcal{Q})$ is presentable, the inclusion $\mathcal{D}^\omega(\mathcal{C}^\omega) \to \mathrm{Def}(\mathcal{C},\mathcal{Q})$ can be extended to a symmetric monoidal, filtered-colimit-preserving functor $\mathrm{Ind}(\mathcal{D}^\omega(\mathcal{C}^\omega)) \to \mathrm{Def}(\mathcal{C},\mathcal{Q})$. We first prove that this functor is an equivalence.

By [Lur09, Proposition 5.3.5.11], it suffices to show that all objects in $\mathcal{D}^\omega(\mathcal{C}^\omega)$ are compact in $\mathrm{Def}(\mathcal{C},\mathcal{Q})$ and also that $\mathcal{D}^\omega(\mathcal{C}^\omega)$ generates $\mathrm{Def}(\mathcal{C},\mathcal{Q})$ under filtered colimits.

We have shown in the proof of Proposition 2.24 that $\{\nu(P) : P \in \mathcal{C}^\omega\}$ are compact, so all objects in $\mathcal{D}^\omega(\mathcal{C}^\omega)$ are compact as compact objects in a stable ∞-category are closed under finite colimits and desuspensions. Moreover, $\mathrm{Def}(\mathcal{C},\mathcal{Q})$ is generated by desuspensions of $\{\nu(P) : P \in \mathcal{C}^\omega\}$ under colimits, but any colimit is a filtered colimit of finite colimits, so we get $\mathrm{Ind}(\mathcal{D}^\omega(\mathcal{C}^\omega)) \simeq \mathrm{Def}(\mathcal{C},\mathcal{Q})$.

By definition, we already have the equivalences $\nu \simeq \overline{\nu}$ and $\tau^{-1} \simeq \overline{\tau}^{-1}$ when restricted to $\mathcal{C}^\omega$ and $\mathcal{D}^\omega(\mathcal{C}^\omega)$. Then it suffices to show that both ν and τ^{-1} preserve filtered colimits, since $\overline{\nu}$ and $\overline{\tau}^{-1}$ are the unique filtered-colimit-preserving extensions. This follows from Proposition 2.12 and the fact that τ^{-1} is a left adjoint. □

4. G-homotopy fixed point spectral sequences

In this section, we apply the general construction of deformation categories to the ∞-category $\mathcal{S}p_G$ of genuine G-spectra. In particular, we show that we can read off $\mathbb{Z}$-graded Mackey-functor-valued homotopy fixed point spectral sequences in $\mathcal{S}p_G$ from the deformation category.

4.1. **Construction of synthetic G-spectra.** Let $\mathbb{F}_G$ be the 1-category of finite G-sets and G-equivariant maps, and let $\mathscr{B}_G := \mathrm{Span}(\mathbb{F}_G)$ be the G-Burnside category, namely, the ∞-span category of the category of finite G-sets.

Roughly speaking, $\mathscr{B}_G$ is (the nerve of) the $(2,1)$-category in which

(1) objects are finite G-sets;
(2) 1-morphisms between two finite G-sets X and Y are spans in $\mathbb{F}_G$

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

and the composite of two spans is defined as the pull-back of spans;
(3) 2-morphisms are isomorphisms of spans.

In particular, $\mathscr{B}_G$ is semi-additive: both finite products and finite coproducts in $\mathscr{B}_G$ are computed by the disjoint unions of the underlying finite G-sets. The Cartesian product of finite G-sets determines a symmetric monoidal structure on $\mathscr{B}_G$.

According to [Bar17], we define the category of spectral Mackey functors

$$P_{\Sigma}^{\mathcal{S}p}(\mathscr{B}_G)$$

as the category of spherical presheaves of spectra on $\mathscr{B}_G$ and equip it with a symmetric monoidal structure through Day convolution. Then by [GM24, Nar16], whenever G is finite, we have an equivalence between presentably symmetric monoidal stable ∞-categories

$$\mathcal{S}p_G \simeq P_{\Sigma}^{\mathcal{S}p}(\mathscr{B}_G).$$

Here, $\mathcal{S}p_G$ is the ∞-category associated to the model category of orthogonal G-spectra. A summary of spectral Mackey functors and equivariant homotopy theory can be found in [CMNN24].

In particular, $\mathcal{S}p_G \simeq P_{\Sigma}^{\mathcal{S}p}(\mathscr{B}_G)$ is rigidly generated by [Ram24, Example 4.40].

Warning 4.1. Note that $\mathscr{B}_G$ is only semi-additive; each mapping space in $\mathscr{B}_G$ is canonically an $\mathbb{E}_\infty$-monoid in $\mathcal{S}$ but not necessarily an $\mathbb{E}_\infty$-group. Then the Yoneda embedding $\mathscr{B}_G \to \mathcal{S}p_G$ is not fully faithful. Instead, it induces a group completion on each mapping space.

Notations 4.2. For any finite G-set $T \in \mathbb{F}_G$, by abusing the notation, we denote the corresponding object in $\mathscr{B}_G$ also by T, and denote its image in $\mathcal{S}p_G$ through the Yoneda embedding by $\Sigma_+^\infty T$. Hence, by Σ_+^∞ we mean one of the following two functors, depending on the context: the stabilization functor $\mathbb{F}_G \to \mathcal{S}p_G$ or the Yoneda embedding $\mathscr{B}_G \to \mathcal{S}p_G$.

Moreover, for any finite G-set $T \in \mathscr{B}_G$, the diagonal map

$$T \to T \times T$$

induces an $\mathbb{E}_\infty$-algebra structure on $T \in \mathscr{B}_G$ contravariantly, and therefore an $\mathbb{E}_\infty$-algebra structure on $\Sigma_+^\infty T \in \mathcal{S}p_G$ since the Yoneda embedding is symmetric monoidal. To be more precise, by the Wirthmüller isomorphism, $\Sigma_+^\infty T$ is self-dual in $\mathcal{S}p_G$, so the $\mathbb{E}_\infty$-coalgebra structure induced by the diagonal further gives rise to an $\mathbb{E}_\infty$-algebra structure since taking dual is contravariant.

Definition 4.3. Let $\mathbb{S}[G] \in \mathrm{CAlg}(\mathcal{S}p_G)$ be the $\mathbb{E}_\infty$-algebra in $\mathcal{S}p_G$ with underlying G-spectra $\Sigma_+^\infty G$ whose $\mathbb{E}_\infty$ structure is obtained through the construction described above.

Now we apply the machine introduced in Section 2 to $\mathcal{S}p_G$.

Construction 4.4. Let $\mathcal{C} := \mathcal{S}p_G$ be the ∞-category of G-spectra, $\tilde{\mathcal{C}} := \mathcal{S}p_G^\omega$ be the full subcategory of $\mathcal{S}p_G$ spanned by all compact objects, and $\mathcal{Q}$ be the collection of $\mathbb{S}[G]$-*split maps* in $\mathcal{S}p_G^\omega$; that is, a map f in $\mathcal{S}p_G^\omega$ is contained in $\mathcal{Q}$ if and only if $\mathbb{S}[G] \otimes f$ admits a section.

Lemma 4.5. *The triple* $(\mathcal{S}p_G, \mathcal{S}p_G^\omega, \mathcal{Q} = \{\mathbb{S}[G]\text{-}split\ maps\})$ *satisfies the conditions in Assumptions 2.1 and 3.12.*

Proof. Firstly, $\mathcal{S}p_G$ is a rigidly generated presentably symmetric monoidal stable ∞-category, so $\mathcal{S}p_G^\omega$ satisfies the requirements on $\tilde{\mathcal{C}}$ in Assumptions 2.1.

Now we show that the class of $\mathbb{S}[G]$-split morphisms satisfies the conditions on $\mathcal{Q}$. It is obvious that the class of $\mathbb{S}[G]$-split morphisms contains all equivalences and is closed under composition and tensor products, so it remains to show that it is closed under pull-backs because $\mathcal{S}p_G^\omega$ is closed under finite limits. Indeed, let $f : X \to Y$ be a $\mathbb{S}[G]$-split map with $s : \mathbb{S}[G] \otimes Y \to \mathbb{S}[G] \otimes X$ a section of $\mathbb{S}[G] \otimes f$ and let $g : Z \to Y$ be any morphism in $\mathcal{S}p_G^\omega$. Then the homotopy commutative diagram

$$\begin{tikzcd}
\mathbb{S}[G] \otimes Z \arrow[r, "s\circ(\mathbb{S}[G]\otimes g)"] \arrow[d, "id"] & \mathbb{S}[G] \otimes X \arrow[d, "\mathbb{S}[G]\otimes f"] \\
\mathbb{S}[G] \otimes Z \arrow[r, "\mathbb{S}[G]\otimes g"] & \mathbb{S}[G] \otimes Y
\end{tikzcd}$$

induces a canonical map $\mathbb{S}[G] \otimes Z \to \mathbb{S}[G] \otimes (X \times_Y Z)$, which makes the pull-back $X \times_Y Z \to Z$ $\mathbb{S}[G]$-split. This proves that Assumptions 2.1 holds.

To verify that the conditions in Assumptions 3.12 are satisfied, it remains to show that $\mathcal{Q}$ is an epimorphism class on $\mathcal{S}p_G^\omega$ with enough injectives.

For any $f : X \to Y$ and $g : Y \to Z$, a section of $(g \circ f) \otimes \mathbb{S}[G] : X \otimes \mathbb{S}[G] \to Z \otimes \mathbb{S}[G]$ determines a section of $g \otimes \mathbb{S}[G]$ by postcomposing with $f \otimes \mathbb{S}[G]$. So $g \circ f \in \mathcal{Q}$ implies $g \in \mathcal{Q}$. Moreover, regarding suspension and desuspension as smashing with $\mathbb{S}^1$ and $\mathbb{S}^{-1}$ respectively, we have $f : X \to Y$ is $\mathbb{S}[G]$-split if and only if Σf is. This shows that the $\mathbb{S}[G]$-split class $\mathcal{Q}$ is an epimorphism class.

Recall that $\mathbb{S}[G]$ admits an $\mathbb{E}_\infty$-algebra structure in $\mathcal{S}p_G$ as shown in Definition 4.3. To show that $\mathcal{S}p_G^\omega$ has enough $\mathcal{Q}$-injectives, we claim that for all $X \in \mathcal{S}p_G^\omega$, tensoring with the unit of $\mathbb{S}[G]$ provides a $\mathcal{Q}$-monic map into an $\mathcal{Q}$-injective

$$X \to \mathbb{S}[G] \otimes X.$$

After tensoring with $\mathbb{S}[G]$, the map

$$\mathbb{S}[G] \otimes X \to \mathbb{S}[G] \otimes \mathbb{S}[G] \otimes X$$

admits a retract given by the multiplication on $\mathbb{S}[G]$, so its cofiber admits a section. This proves that $X \to \mathbb{S}[G] \otimes X$ is $\mathcal{Q}$-monic.

To see $\mathbb{S}[G] \otimes X$ is an $\mathcal{Q}$-injective, we need to show that given a $\mathcal{Q}$-monic map $f : Y \to Z$, any map $Y \to \mathbb{S}[G] \otimes X$ can be extended along $f : Y \to Z$. Since f is $\mathcal{Q}$-monic, $\mathbb{S}[G] \otimes f$ admits a retract $r : \mathbb{S}[G] \otimes Z \to \mathbb{S}[G] \otimes Y$. By the free-forgetful adjunction between $\mathcal{S}p_G^\omega$ and the category of $\mathbb{S}[G]$-modules, we can always extend $Y \to \mathbb{S}[G] \otimes X$ along the unit $Y \to \mathbb{S}[G] \otimes Y$. Let $g : \mathbb{S}[G] \otimes Y \to \mathbb{S}[G] \otimes X$ be the resulting extension. Then the following commutative diagram provides the desired extension along $f : Y \to Z$.

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & Z \arrow[d] & & \\ \mathbb{S}[G] \otimes Y \arrow[r, "\mathbb{S}[G]\otimes f"] & \mathbb{S}[G] \otimes Z \arrow[r, "r"] & \mathbb{S}[G] \otimes Y \arrow[r, "g"] & \mathbb{S}[G] \otimes X \end{tikzcd}$$

This ends the proof. $\square$

Therefore, we can construct a deformation category with the generic fiber $\mathcal{S}p_G$ through Definition 2.4.

Definition 4.6. We define the ∞-category $\mathcal{S}yn_{\mathbb{S}[G]}$ of $\mathbb{S}[G]$-*synthetic spectra* to be the deformation category $\mathrm{Def}(\mathcal{S}p_G, \mathcal{Q}) \simeq Sh_{\Sigma}^{\mathcal{S}p}(\mathcal{S}p_G^{\omega})$.

Remark 4.7. Since the collection of $\mathbb{S}[G]$-split maps satisfies Assumptions 3.12, the category of $\mathbb{S}[G]$-synthetic spectra $\mathcal{S}yn_{\mathbb{S}[G]}$ is equivalent to the Ind-completion of the perfect derived ∞-category $\mathcal{D}^{\omega}(\mathcal{S}p_G^{\omega})$.

In order to compare $\mathbb{S}[G]$-synthetic spectra with filtered G-spectra, we need to further verify Assumption 2.35.

Proposition 4.8. *There exists a unique symmetric monoidal left adjoint c extending*

$$\nu \circ \Sigma_+^{\infty} : \mathscr{B}_G \to \mathcal{S}p_G \to \mathcal{S}yn_{\mathbb{S}[G]}$$

along the Yoneda embedding $\mathscr{B}_G \to P_{\Sigma}^{\mathcal{S}p}(\mathscr{B}_G)$. That is, we have the following commutative diagram:

$$\begin{tikzcd} \mathscr{B}_G \arrow[r, "\nu\circ\Sigma_+^{\infty}"] \arrow[d, "\Sigma_+^{\infty}"] & \mathcal{S}yn_{\mathbb{S}[G]} \\ \mathcal{S}p_G \simeq P_{\Sigma}^{\mathcal{S}p}(\mathscr{B}_G) \arrow[ur, dashed, "c"] & \end{tikzcd}$$

Moreover, we have a commutative diagram of symmetric monoidal left adjoints between presentably symmetric monoidal stable ∞-categories

$$\begin{tikzcd} & \mathcal{S}yn_{\mathbb{S}[G]} \arrow[dr, "\tau^{-1}"] & \\ \mathcal{S}p_G \arrow[ur, "c"] \arrow[rr, equal] & & \mathcal{S}p_G \end{tikzcd}$$

Equivalently, c satisfies Assumption 2.35.

Proof. Recall that $P(\mathscr{B}_G)$ is obtained from $\mathscr{B}_G$ by freely adjoining all colimits. Consider the induced colimit-preserving extension

$$\overline{F} : P(\mathscr{B}_G) \to \mathcal{S}yn_{\mathbb{S}[G]}$$

of $\nu|_{\mathscr{B}_G}$. This functor $\overline{F}$ admits a right adjoint $\overline{G}$ since both the source and the target of F are presentable. Hence, we get the adjoint pair

$$\begin{tikzcd} P(\mathscr{B}_G) \arrow[r, shift left, "\overline{F}"] & \mathcal{S}yn_{\mathbb{S}[G]} \arrow[l, shift left, "\overline{G}"] \end{tikzcd}.$$

Note that $\nu \circ \Sigma^\infty_+$ preserves finite coproducts since both ν and Σ^∞_+ do. Then by [Lur09, Proposition 5.5.8.10], $\overline{G}$ factors through $P_\Sigma(\mathscr{B}_G)$. Restricting the above adjoint pair to $P_\Sigma(\mathscr{B}_G)$, we get another adjoint pair

$$P_\Sigma(\mathscr{B}_G) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{S}yn_{\mathbb{S}[G]}.$$

Since $\Sigma^\infty_+ T$ is compact in $\mathcal{S}p_G$ for any finite G-set T and ν is symmetric monoidal when restricted to compact objects, it follows that $\nu \circ \Sigma^\infty_+$ is symmetric monoidal. Then, by the universal property of Day convolution, both $\overline{F}$ and F are symmetric monoidal. Moreover, by the adjoint pair

$$\mathcal{P}r^{\mathrm{L}} \overset{-\otimes \mathcal{S}p}{\rightleftarrows} \mathcal{P}r^{\mathrm{L}}_{\mathrm{st}}$$

with a symmetric monoidal left adjoint, the symmetric monoidal left adjoint F induces a symmetric monoidal left adjoint

$$\mathcal{S}p_G \simeq P_\Sigma(\mathscr{B}_G) \otimes \mathcal{S}p \to \mathcal{S}yn_{\mathbb{S}[G]} \in \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{st}})$$

which is precisely the desired functor c.

Note that the colimit-preserving extension of $\nu \circ \Sigma^\infty_+$ to the essential image of

$$\Sigma^\infty_+ : \mathscr{B}_G \to \mathcal{S}p_G$$

is unique as $\mathcal{S}p_G$ is additive. Then the uniqueness of c follows, since the desuspensions of $\{\Sigma^\infty_+ T : T \in \mathscr{B}_G\}$ generate $\mathcal{S}p_G$ under colimits.

To show $\tau^{-1}c = id$, note that both τ^{-1} and c preserve colimits and $\tau^{-1}c = \tau^{-1}\nu = id$ hold on $\mathscr{B}_G$. Then the proposition follows. □

Remark 4.9. The $\mathcal{S}p_G$-enrichment on $\mathcal{S}yn_{\mathbb{S}[G]}$ along $c : \mathcal{S}p_G \to \mathcal{S}yn_{\mathbb{S}[G]}$ can be described in detail as follows. For any two $\mathbb{S}[G]$-synthetic spectra $X, Y \in \mathcal{S}yn_{\mathbb{S}[G]}$, regarding $\mathcal{S}p_G$ as $P^{\mathcal{S}p}_\Sigma(\mathscr{B}_G)$, we have $\mathrm{F_G}(X, Y)$ is the presheaf sending a finite G-set $T \in \mathscr{B}_G$ to

$$\mathrm{F}(T, \mathrm{F_G}(X, Y)) \simeq \mathrm{F}(c(T) \otimes X, Y) \simeq \mathrm{F}(\nu(T) \otimes X, Y).$$

Remark 4.10. By Construction 2.38, we have an adjoint pair

$$\mathrm{Fil}(\mathcal{S}p_G) \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{S}yn_{\mathbb{S}[G]}$$

where the right adjoint is computed by

$$i_* X \simeq \mathrm{F_G}(\nu\mathbb{S}, \Sigma^{0,-*}X).$$

More precisely, for any $T \in \mathscr{B}_G$, the evaluation on T is given by

$$i_* X(T) \simeq \mathrm{F}(\nu T, \Sigma^{0,-*}X).$$

As a corollary of Theorem 2.39, we have the following result.

Corollary 4.11. *Let the category of cellular $\mathbb{S}[G]$-synthetic spectra $\mathcal{S}yn^{\mathrm{cell}}_{\mathbb{S}[G]}$ be the full subcategory of $\mathcal{S}yn_{\mathbb{S}[G]}$ generated by*

$$\{\Sigma^{m,n}\nu T : m, n \in \mathbb{Z}, T \in \mathscr{B}_G\}$$

under colimits. Then the restriction of the above adjoint pair on $\mathcal{S}yn^{\mathrm{cell}}_{\mathbb{S}[G]}$ induces an equivalence

$$\mathcal{S}yn^{\mathrm{cell}}_{\mathbb{S}[G]} \simeq \mathcal{M}od_{i_*\nu\mathbb{S}}(\mathrm{Fil}(\mathcal{S}p_G)).$$

4.2. **Recollections on homotopy fixed point spectral sequences.** Recall that given a G-spectrum $X \in \mathcal{S}p_G$, the homotopy fixed point G-spectrum of X is defined to be its Borel completion, which is the internal hom

$$\mathrm{F}_{\mathrm{G}}(\Sigma^\infty_+ EG, X) \in \mathcal{S}p_G.$$

Here, EG is the total space of the universal G-bundle, which is also the unique (up to homotopy) G-space with a free G-action and contractible underlying space.

We call $\mathrm{F}_{\mathrm{G}}(\Sigma^\infty_+ EG, X)$ the homotopy fixed point of X since we have the equivalence

$$\mathrm{F}(\Sigma^\infty_+ G, \mathrm{F}_{\mathrm{G}}(\Sigma^\infty_+ EG, X)) \simeq \mathrm{F}(\Sigma^\infty_+ G, X) \in \mathcal{S}p^{BG}$$

between two Borel G-spectra and for any subgroup $H \subset G$, the categorical fixed point

$$\mathrm{F}_{\mathrm{G}}(\Sigma^\infty_+ EG, X)^H \simeq \mathrm{F}(\Sigma^\infty_+ G, X)^{hH}$$

coincides with the H-homotopy fixed point of $\mathrm{F}(\Sigma^\infty_+ G, X)$.

Moreover, since the G-space EG is usually defined as the geometric realization of the two-side Bar construction $B(G, G, *)$ and the left adjoint Σ^∞_+ is symmetric monoidal and preserves geometric realizations, $\Sigma^\infty_+ EG$ is equivalent to the geometric realization of the simplicial G-spectrum $B_\bullet(\Sigma^\infty_+ G, \Sigma^\infty_+ G, *)$. Therefore, we can regard the homotopy fixed point G-spectrum of X as the totalization of the cosimplicial G-spectrum $\mathrm{F}_{\mathrm{G}}(B_\bullet(\Sigma^\infty_+ G, \Sigma^\infty_+ G, *), X)$.

Definition 4.12. The homotopy fixed point spectral sequence of X is defined to be the Bousfield-Kan spectral sequence, see [BK72], of the cosimplicial G-spectrum

$$\mathrm{F}_{\mathrm{G}}(B_\bullet(\Sigma^\infty_+ G, \Sigma^\infty_+ G, *), X).$$

Equivalently, the homotopy fixed point spectral sequence of X is the spectral sequence induced by the coskeleton filtration

$$\mathrm{Tot}_n \, \mathrm{F}_{\mathrm{G}}(B_\bullet(\Sigma^\infty_+ G, \Sigma^\infty_+ G, *), X) := \lim_{\Delta_{\leqslant n}} \mathrm{F}_{\mathrm{G}}(B_\bullet(\Sigma^\infty_+ G, \Sigma^\infty_+ G, *), X)$$

on $\mathrm{F}_{\mathrm{G}}(\Sigma^\infty_+ EG, X) \simeq \mathrm{Tot}\, \mathrm{F}_{\mathrm{G}}(B_\bullet(\Sigma^\infty_+ G, \Sigma^\infty_+ G, *), X)$.

Theorem 4.13. *For any $X \in \mathcal{S}p_G$, the construction of the cosimplicial G-spectrum defined in Definition 4.12 is equivalent to tensoring with the cobar construction of $\mathbb{S}[G]$:*

$$\mathrm{F}_{\mathrm{G}}(B_\bullet(\Sigma^\infty_+ G, \Sigma^\infty_+ G, *), X) \simeq \mathbb{S}[G]^{\bullet+1} \otimes X.$$

In particular, the homotopy fixed point spectral sequence of X is isomorphic to the $\mathbb{S}[G]$-Adams spectral sequence of X.

Proof. Recall that the Bar construction $B_\bullet(\Sigma^\infty_+ G, \Sigma^\infty_+ G, *)$ is defined as

$$\Sigma^\infty_+ G \leftleftarrows\!\!\!\!\to \Sigma^\infty_+ (G \times G) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow} \cdots$$

Here, $B_\bullet(\Sigma^\infty_+ G, \Sigma^\infty_+ G, *) \simeq \Sigma^\infty_+ G^{\bullet+1}$, the degeneracy maps are diagonals and the face maps are projections.

Since G is a finite group, each $\Sigma^\infty_+ G^{n+1}$ is a finite coproduct of $\Sigma^\infty_+ G$ and thus is dualizable. So on the left-hand side, we have

$$\mathrm{F}_{\mathrm{G}}(B_\bullet(\Sigma^\infty_+ G, \Sigma^\infty_+ G, *), X) \simeq \mathrm{F}_{\mathrm{G}}(\Sigma^\infty_+ G^{\bullet+1}, \mathbb{S}) \otimes X.$$

Then it suffices to prove the theorem for $X = \mathbb{S}$.

By the Wirthmüller isomorphism,

$$\mathrm{F}_{\mathrm{G}}(\Sigma^\infty_+ G^{\bullet+1}, \mathbb{S}) \simeq \Sigma^\infty_+ G^{\bullet+1} \simeq \mathbb{S}[G]^{\bullet+1}.$$

As shown in Definition 4.3, the multiplication on $\mathbb{S}[G]$, and thus the codegeneracy maps in the cobar construction $\mathbb{S}[G]^{\bullet+1}$, is induced by the diagonal on G through $\mathrm{F}_{\mathrm{G}}(\Sigma^\infty_+ G, \mathbb{S})$; the unit on $\mathbb{S}[G]$, and thus the coface maps in $\mathbb{S}[G]^{\bullet+1}$, is induced by the projection $G \to *$ on $\mathrm{F}_{\mathrm{G}}(\Sigma^\infty_+ G, \mathbb{S})$. Hence, the two cosimplicial objects coincide when $X = \mathbb{S}$. This ends the proof. $\square$

Remark 4.14. A similar statement can be found in [MNN17, AKBBK23].

However, the coskeleton filtration on $\mathrm{F}_{\mathrm{G}}(\Sigma^\infty_+ EG, X) \simeq \mathrm{Tot}\, \mathbb{S}[G]^{\bullet+1} \otimes X$

$$\cdots \to \mathrm{Tot}_{n+1}\, \mathbb{S}[G]^{\bullet+1} \otimes X \to \mathrm{Tot}_n\, \mathbb{S}[G]^{\bullet+1} \otimes X \to \cdots \to \mathrm{Tot}_0\, \mathbb{S}[G]^{\bullet+1} \to 0 \to \cdots$$

is an increasing filtration: it has a trivial sequential colimit, but its sequential limit is $\mathrm{Tot}\, \mathbb{S}[G]^{\bullet+1} \otimes X$.

As pointed out in Remark A.7, it would be more general to consider decreasing filtration on $\mathrm{Tot}\, \mathbb{S}[G]^{\bullet+1} \otimes X$. We will construct a decreasing filtration associated to the above coskeleton tower following the classical construction of the Adams spectral sequences.

Let $\overline{\mathbb{S}[G]}$ be the fiber of the unit $\mathbb{S} \to \mathbb{S}[G]$. Then by [MNN17, Proposition 2.14], we have

$$\mathrm{cofib}(\overline{\mathbb{S}[G]}^{\otimes n+1} \to \mathbb{S}) \simeq \mathrm{Tot}_n(\mathbb{S}[G]^{\bullet+1}).$$

Consider the tower of cofiber sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \overline{\mathbb{S}[G]}^{\otimes n+1} \arrow[r] \arrow[d] & \overline{\mathbb{S}[G]}^{\otimes n} \arrow[r] \arrow[d] & \cdots \arrow[r] & \overline{\mathbb{S}[G]} \arrow[r] \arrow[d] & \mathbb{S} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbb{S} \arrow[r] \arrow[d] & \mathbb{S} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathbb{S} \arrow[r] \arrow[d] & \mathbb{S} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{Tot}_n(\mathbb{S}[G]^{\bullet+1}) \arrow[r] & \mathrm{Tot}_{n-1}(\mathbb{S}[G]^{\bullet+1}) \arrow[r] & \cdots \arrow[r] & \mathrm{Tot}_0(\mathbb{S}[G]^{\bullet+1}) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Note that after completing each horizontal filtered object, we still get a cofiber sequence of filtered objects. However, the completion of the constant filtration on $\mathbb{S}$ is trivial, so the top and bottom filtered objects have equivalent completion up to a suspension. Therefore, the top and bottom filtered objects induce isomorphic spectral sequences up to regrading.

Definition 4.15. Let $X \in \mathcal{S}p_G$ be any G-spectrum. We define the $\mathbb{S}[G]$-Adams filtration, and also the homotopy fixed point filtration, of X as the following filtered G-spectrum

$$AF_*(X) : \cdots \to \overline{\mathbb{S}[G]}^{\otimes n+1} \otimes X \to \overline{\mathbb{S}[G]}^{\otimes n} \otimes X \to \cdots \to \overline{\mathbb{S}[G]} \otimes X \to X \to X \to \cdots.$$

More precisely,

$$AF_n(X) := \begin{cases} \overline{\mathbb{S}[G]}^{\otimes n} \otimes X, & \text{if } n > 0; \\ X, & \text{if } n \leqslant 0. \end{cases}$$

Remark 4.16. The completion of the $\mathbb{S}[G]$-Adams filtration $AF_*(X)$ is precisely the modified Dold-Kan construction of the cosimplicial object $\mathbb{S}[G]^{\bullet+1} \otimes X$.

4.3. **τ-Bockstein spectral sequences.** In this section, we show the equivalence between the τ-Bockstein spectral sequences in $\mathcal{S}yn_{\mathbb{S}[G]}$ and the $\mathbb{S}[G]$-Adams spectral sequences in $\mathcal{S}p_G$, and hence also the G-homotopy fixed point spectral sequences by Theorem 4.13.

Definition 4.17. Let $X \in \mathcal{S}p_G$ be any G-spectrum. The τ-Bockstein filtration on X is defined to be the filtered $\mathbb{S}[G]$-synthetic spectra

$$Y(X)_{\geqslant *} = (\cdots \xrightarrow{\tau} \Sigma^{0,n}\nu X \xrightarrow{\tau} \Sigma^{0,n+1}\nu X \xrightarrow{\tau} \cdots) \in \mathrm{Fil}(\mathcal{S}yn_{\mathbb{S}[G]}),$$

which coincides with the Whitehead tower of $Y(X)$ by Proposition 2.25.

Now we state the main theorem of this section. Roughly speaking, the following theorem implies that the τ-Bockstein spectral sequences is the décalage of the homotopy fixed point spectral sequences, so they are equivalent up to page-turning.

Theorem 4.18. *Let $X \in \mathcal{S}p_G$ be any G-spectrum and $P \in \mathcal{S}p_G^\omega$ be a compact G-spectrum. Then we have the following equivalence in the category of filtered spectra:*

$$\mathrm{F}(\nu(P), Y(X)_{\geqslant *}) \simeq \mathrm{Déc}\,\mathrm{F}(P, AF_*(X)) \in \mathrm{Fil}(\mathcal{S}p).$$

Here, $\mathrm{Déc} : \mathrm{Fil}(\mathcal{S}p) \to \mathrm{Fil}(\mathcal{S}p)$ *is defined using the standard t-structure on $\mathcal{S}p$.*

Proof. For a fixed $P \in \mathcal{S}p_G^\omega$, we first claim that both sides of the above equivalence preserve filtered colimits on X as functors from $\mathcal{S}p$ to $\mathrm{Fil}(\mathcal{S}p)$.

For $\mathrm{F}(\nu(P), Y(-)_{\geqslant *})$, note that this is a composition of the spectral Yoneda embedding Y, the Whitehead tower $(-)_{\geqslant *}$, and $\mathrm{F}(\nu(P), -)$, each of which preserves filtered colimits by Theorem 2.26, the fact that the t-structure on $\mathcal{S}yn_{\mathbb{S}[G]}$ is compatible with filtered colimits, and the compactness of $\nu(P)$.

Similarly, $\mathrm{Déc}\,\mathrm{F}(P, AF_*(-))$ is a composition of $\mathbb{S}[G]$-Adams filtration AF_*, $\mathrm{F}(P, -)$, and Déc, each of which preserves filtered colimits by the fact that AF_* is smashing, the compactness of P, and Proposition A.15. Therefore, it suffices to prove the equivalence for $X \in \mathcal{S}p_G^\omega$.

Recall that in Lemma 4.5, we have shown that the collection of $\mathbb{S}[G]$-split maps $\mathcal{Q}$ is an epimorphism class with enough injectives. Let $X \in \mathcal{S}p_G^\omega$ be $\mathcal{Q}$-injective. We claim that in this case, the spherical presheaf of spectra $\mathrm{F}(-, X)_{\geqslant n}$ is already a sheaf for any integer n.

By Theorem 2.13, it suffices to check that for any fiber sequence

$$F \to A \to B$$

in $\mathcal{S}p_G^\omega$ with $\{A \to B\} \in \mathcal{Q}$,

$$\mathrm{F}(B, X)_{\geqslant n} \to \mathrm{F}(A, X)_{\geqslant n} \to \mathrm{F}(F, X)_{\geqslant n}$$

is a fiber sequence in $\mathcal{S}p$.

Since $\mathcal{Q}$ is closed under suspension and desuspension, $\mathcal{Q}$-injectives are also closed under suspension and desuspension, so it suffices to check this for $n = 0$. In this case,

$$\mathrm{F}(B, X)_{\geqslant 0} \to \mathrm{F}(A, X)_{\geqslant 0} \to \mathrm{F}(F, X)_{\geqslant 0}$$

is a fiber sequence if and only if $[A, X] \to [F, X]$ is an epimorphism in $\mathcal{A}b$. This holds by the definition of $\mathcal{Q}$-injectives because $F \to A$ is $\mathcal{Q}$-monic.

Therefore, since sheafification is t-exact, the τ-Bockstein filtration of an $\mathcal{Q}$-injective compact G-spectrum X is precisely the tower of sheaves

$$\cdots \to \mathrm{F}(-, X)_{\geqslant n+1} \to \mathrm{F}(-, X)_{\geqslant n} \to \mathrm{F}(-, X)_{\geqslant n-1} \to \cdots$$

Since the connective cover is computed level-wise for $Y(X)$ whenever X is $\mathcal{Q}$-injective, we can further compute the mapping spectra $\mathrm{F}(\nu P, Y(X)_{\geqslant n})$ as

$$\begin{aligned}
&\Omega^{\infty-m}\,\mathrm{F}(\nu P, Y(X)_{\geqslant n})\\
\simeq\,&\mathrm{map}(\nu P, \Sigma^{m,-m}(Y(X)_{\geqslant n}))\\
\simeq\,&\mathrm{map}(\nu P, Y(X)_{\geqslant n+m})\\
\simeq\,&\mathrm{map}_{(\mathcal{S}yn_{\mathbb{S}[G]})_{\geqslant 0}}(\nu P, Y(X)_{\geqslant \max\{n+m,0\}})\\
\simeq\,&\Omega^{\infty}\,\mathrm{F}(P,X)_{\geqslant \max\{n+m,0\}}\\
\simeq\,&\Omega^{\infty}\,\mathrm{F}(P,X)_{\geqslant n+m}\\
\simeq\,&\Omega^{\infty-m}\,\mathrm{F}(P,X)_{\geqslant n}.
\end{aligned}$$

Therefore, $\mathrm{F}(\nu(P), Y(X)_{\geqslant *})$ is precisely the Whitehead tower of $\mathrm{F}(P,X)$ in spectra

$$\cdots \to \mathrm{F}(P,X)_{\geqslant n+1} \to \mathrm{F}(P,X)_{\geqslant n} \to \mathrm{F}(P,X)_{\geqslant n-1} \to \cdots$$

whenever X is $\mathcal{Q}$-injective.

For general compact G-spectrum $X \in \mathcal{S}p_G^{\omega}$, consider the augmented cosimplicial G-spectrum

$$X \to \mathbb{S}[G] \otimes X \rightrightarrows \mathbb{S}[G] \otimes \mathbb{S}[G] \otimes X \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots .$$

Applying $\mathrm{F}(\nu(P), Y(-)_{\geqslant *})$ to the above diagram, we get an augmented cosimplicial filtered spectrum

$$\mathrm{F}(\nu(P), Y(X)_{\geqslant *}) \to \mathrm{F}(\nu(P), Y(\mathbb{S}[G] \otimes X)_{\geqslant *}) \rightrightarrows \cdots .$$

We claim that the induced morphism of filtered spectra

$$\mathrm{F}(\nu(P), Y(X)_{\geqslant *}) \to \mathrm{Tot}(\mathrm{F}(\nu(P), Y(\mathbb{S}[G]^{\otimes \bullet+1} \otimes X)_{\geqslant *}))$$

is an equivalence up to completion.

Equivalently, it suffices to show this map is an equivalence on associated graded spectra by Proposition A.9. Therefore, we only need to show the augmented cosimplicial object

$$\pi_n^{\heartsuit} Y(X) \to \pi_n^{\heartsuit} Y(\mathbb{S}[G] \otimes X) \rightrightarrows \pi_n^{\heartsuit} Y(\mathbb{S}[G] \otimes \mathbb{S}[G] \otimes X) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots .$$

is a limit diagram in $\mathcal{S}yn_{\mathbb{S}[G]}$ since $\mathrm{F}(\nu(P), -)$ commutes with taking assocoated graded object. Here, we regard the t-structure n-th homotopy group as the functor

$$\pi_n^{\heartsuit} : \mathcal{S}yn_{\mathbb{S}[G]} \to \mathcal{S}yn_{\mathbb{S}[G]}^{\heartsuit} \to \mathcal{S}yn_{\mathbb{S}[G]}$$

where the last functor is the inclusion.

Since $Y(X)$ is τ-invertible and

$$\pi_n^{\heartsuit}(\Sigma^{0,-1} X) \cong \pi_{n-1}^{\heartsuit} X,$$

we have τ induces an isomorphism between $\pi_0^{\heartsuit} Y(X)$ and $\pi_{n-1}^{\heartsuit} Y(X)$. Thus, it suffices to show

$$\pi_n^{\heartsuit} \nu(X) \simeq C\tau \otimes \nu(X) \simeq \mathrm{Tot}(C\tau \otimes \nu(\mathbb{S}[G]^{\otimes \bullet+1} \otimes X)).$$

Note that each term in the augmented cosimplicial object is coconnective, and $(\mathcal{S}yn_{\mathbb{S}[G]})_{\leqslant 0}$ is closed under all limits. Hence, we only need to show that the above is an equivalence on t-structure homotopy groups since the t-structure is right complete. Also, we can further assume $X = \mathbb{S}$ since X is dualizable and thus $- \otimes \nu(X)$ preserves all limits.

We prove the claim by computing the Bousfield-Kan spectral sequence for t-structure homotopy groups. By [PP23, Proposition 5.34], since the unit $u : \mathbb{S} \to \mathbb{S}[G]$ is $\mathcal{Q}$-monic, ν sends the cofiber sequence

$$\mathbb{S} \xrightarrow{u} \mathbb{S}[G] \xrightarrow{\delta} Cu$$

to a cofiber sequence

$$\nu\mathbb{S} \xrightarrow{u} \nu\mathbb{S}[G] \xrightarrow{\delta} \nu Cu$$

By [MNN17, Proposition 2.14], we can identify it with the spectral sequence of t-structure homotopy groups associated to the filtered object

$$\cdots \to (\Omega\nu Cu)^{\otimes n+1} \otimes C\tau \to (\Omega\nu Cu)^{\otimes n} \otimes C\tau \to \cdots \to (\Omega\nu Cu) \otimes C\tau \to C\tau \to \cdots$$

since $\Omega\nu Cu$ is the fiber of the unit $\nu\mathbb{S} \to \nu(\mathbb{S}[G])$. Its E_1 page is the t-structure homotopy groups of the chain complex

$$\cdots \to 0 \to \nu\mathbb{S}[G] \otimes C\tau \to \nu\mathbb{S}[G] \otimes \nu Cu \otimes C\tau \to \nu(\mathbb{S}[G]) \otimes (\nu Cu)^{\otimes 2} \otimes C\tau \to \cdots$$

and each differential factors through as

$$\begin{tikzcd}
\nu(\mathbb{S}[G]) \otimes (\nu Cu)^{\otimes n} \otimes C\tau \arrow[rr] \arrow[dr, "\nu\delta"'] & & \nu(\mathbb{S}[G]) \otimes (\nu Cu)^{\otimes n+1} \otimes C\tau \\
& \nu Cu \otimes (\nu Cu)^{\otimes n} \otimes C\tau \arrow[ur, "\nu u"'] &
\end{tikzcd}$$

Also by [PP23, Proposition 5.34], we have

$$\nu(Cu^n) \xrightarrow{u} \nu(\mathbb{S}[G] \otimes Cu^n) \xrightarrow{\delta} \nu(Cu^{n+1})$$

induces a short exact sequence on t-structure homotopy group. This implies the above E_1-page is a long exact sequence except for

$$0 \to \pi_0^{\heartsuit}\nu\mathbb{S}[G] \to \pi_0^{\heartsuit}\nu\mathbb{S}[G] \otimes \nu Cu$$

where has a nontrivial kernel $\pi_0^{\heartsuit}(\nu\mathbb{S})$.

So far, we have shown that

$$\mathrm{F}(\nu(P), Y(X)_{\geqslant *}) \to \mathrm{Tot}(\mathrm{F}(P, \mathbb{S}[G]^{\otimes \bullet+1} \otimes X)_{\geqslant *})$$

is an equivalence up to completion.

On the other hand, the completion of $\mathrm{F}(P, AF_*(X))$ is the modified Dold-Kan construction to $\mathrm{F}(P, \mathbb{S}[G]^{\otimes \bullet+1} \otimes X)$ by Remark 4.16, so $\mathrm{D\acute{e}c}\,\mathrm{F}(P, AF_*(X))$ and the décalage of the modified Dold-Kan construction to $\mathrm{F}(P, \mathbb{S}[G]^{\otimes \bullet+1} \otimes X)$ coincide up to completion by Proposition A.14. The latter is also $\mathrm{Tot}(\mathrm{F}(P, \mathbb{S}[G]^{\otimes \bullet+1} \otimes X)_{\geqslant *})$ by Theorem A.19.

Since décalage preserves underlying spectra by Proposition A.15, it remains to check that

$$\mathrm{F}(\nu(P), Y(X)_{\geqslant *}) \to \mathrm{Tot}(\mathrm{F}(P, \mathbb{S}[G]^{\otimes \bullet+1} \otimes X)_{\geqslant *})$$

and

$$\mathrm{F}(P, AF_*(X)) \to \mathrm{Tot}(\mathrm{F}(P, \mathbb{S}[G]^{\otimes \bullet+1} \otimes X)_{\geqslant *})$$

coincide on the underlying spectra. This holds because both of them are

$$\mathrm{F}(\nu(P), Y(X)) \simeq \mathrm{F}(P, X) \to \mathrm{Tot}(\mathrm{F}(P, \mathbb{S}[G]^{\otimes \bullet+1} \otimes X))$$

by the compactness of P. Therefore, the n-th stage of both $\mathrm{F}(\nu(P), Y(X)_{\geqslant *})$ and $\mathrm{F}(P, AF_*(X))$ are the pull-back of

$$\begin{tikzcd} & \mathrm{Tot}(\mathrm{F}(P, \mathbb{S}[G]^{\otimes \bullet+1} \otimes X)_{\geqslant n}) \arrow[d] \\ \mathrm{F}(\nu(P), Y(X)) \arrow[r] & \mathrm{Tot}(\mathrm{F}(P, \mathbb{S}[G]^{\otimes \bullet+1} \otimes X)) \end{tikzcd}$$

This ends the proof. $\square$

As a corollary, we can further get an equivalence between filtered G-spectra.

Corollary 4.19. *Let $i_* : \mathcal{S}yn_{\mathbb{S}[G]} \to \mathrm{Fil}(\mathcal{S}p_G)$ be the right adjoint defined in Construction 2.38. Then for any G-spectra $X \in \mathcal{S}p_G$, we have an equivalence*

$$i_*(\nu X) \simeq \mathrm{Déc} AF_*(X) \in \mathrm{Fil}(\mathcal{S}p_G).$$

Here, Déc : $\mathrm{Fil}(\mathcal{S}p_G) \to \mathrm{Fil}(\mathcal{S}p_G)$ *is defined using the standard t-structure on $\mathcal{S}p_G$ with heart the category of Mackey functors.*

In particular, we have

$$\mathcal{S}yn^{\mathrm{cell}}_{\mathbb{S}[G]} \simeq \mathcal{M}od_{\mathrm{Déc} AF_*(\mathbb{S})}(\mathrm{Fil}(\mathcal{S}p_G)).$$

Proof. By Remark 4.10, for any $T \in \mathrm{Span}(\mathbb{F}_G)$, the evaluation on T of $i_*(\nu X)$ is precisely the filtered spectra

$$\mathrm{F}(\nu(T), \Sigma^{0,-*}\nu X) \simeq \mathrm{F}(\nu(T), Y(X)_{\geqslant *}).$$

Then by Theorem 4.18, we have

$$i_*(\nu X)(T) \simeq \mathrm{Déc}(AF_*(X)(T)) \in \mathrm{Fil}(\mathcal{S}p_G)$$

for all $T \in \mathrm{Span}(\mathbb{F}_G)$.

Therefore, it remains to show the décalage functor commutes with the evaluation on T. This holds automatically by definition, since taking the associated chain complex object from a filtered object, taking the point-wise connective covers of the chain complex, and taking the sequential colimit of a filtered object all commute with the evaluation on T. $\square$

Moreover, Corollary 4.19 implies the monoidality of the homotopy fixed point spectral sequences.

Corollary 4.20. *The décalage of the $\mathbb{S}[G]$-Adams filtration*

$$\mathcal{S}p_G \xrightarrow{AF_*} \mathrm{Fil}(\mathcal{S}p_G) \xrightarrow{\mathrm{Déc}} \mathrm{Fil}(\mathcal{S}p_G)$$

is canonically lax symmetric monoidal. In particular, the $\mathbb{Z}$-graded Mackey-functor-valued homotopy fixed point spectral sequence of an $\mathbb{E}_\infty$-algebra in $\mathcal{S}p_G$ admits a multiplicative structure.

Proof. This follows from Corollary 4.19 because ν is lax symmetric monoidal and i_* is lax symmetric monoidal as the right adjoint to a symmetric monoidal functor. $\square$

Appendix A. Filtered objects and décalage

This appendix briefly reviews the theory of filtered objects and décalage. Décalage was initially introduced by Deligne in [Del71] as a machine of page-turning on a spectral sequence. An ∞-categorical description of décalage was constructed in [Lev15] in a cosimplicial setting, and later, Hedenlund defines décalage as a self-functor on the category of filtered spectra in [Hed20]. Besides the category of spectra, this construction can be generalized to all stable ∞-categories with a (well-behaved) t-structure that admits sequential limits and colimits. A summary of this can be found in [Ant24], in which Antieau proves that this definition of décalage does preserve the associated spectral sequences up to a page-turning and re-grading.

Throughout this section, let $\mathcal{C}$ be a stable ∞-category with a t-structure that admits sequential limits and colimits.

Definition A.1. Let $\mathbb{Z}^{\leqslant}$ be the poset of integers. A filtered object in $\mathcal{C}$ is a $\mathcal{C}$-valued presheaf on $\mathbb{Z}^{\leqslant}$; that is, a functor from $(\mathbb{Z}^{\leqslant})^{op}$ to $\mathcal{C}$. We denote the ∞-category $\mathrm{Fun}((\mathbb{Z}^{\leqslant})^{op}, \mathcal{C})$ of filtered objects in $\mathcal{C}$ by $\mathrm{Fil}(\mathcal{C})$.

Equivalently, a filtered object $X_* \in \mathrm{Fil}(\mathcal{C})$ is a sequence of composable maps

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$$

where X_n is the evaluation of X on $n \in \mathbb{Z}^{\leqslant}$.

Here are several examples of filtered objects.

Example A.2. For any object $X \in \mathcal{C}$, we can construct a filtered object $c(X) \in \mathrm{Fil}(\mathcal{C})$, called the trivial filtration on X, such that

$$c(X)_n = \begin{cases} 0, & \text{if } n > 0; \\ X, & \text{if } n \leqslant 0, \end{cases}$$

and all nontrivial maps are the identity on X; that is, $c(X)$ is the filtered object

$$\cdots \to 0 \to 0 \to X \xrightarrow{id} X \xrightarrow{id} X \xrightarrow{id} \cdots .$$

This determines a functor $c : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ which is, by definition of left Kan extension, the left adjoint to the evaluation Ev_0 on $0 \in \mathbb{Z}^{\leqslant}$.

Example A.3. Similarly, one can also construct a functor $Y : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ sending $X \in \mathcal{C}$ to the constant presheaf with value X; that is, the filtered object

$$\cdots \xrightarrow{id} X \xrightarrow{id} X \xrightarrow{id} X \xrightarrow{id} X \xrightarrow{id} X \xrightarrow{id} \cdots .$$

We call $Y(X)$ the constant filtration on X.

The constant filtration functor $Y : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ is right adjoint to the sequential colimit functor $\mathrm{colim} : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$ and left adjoint to the sequential colimit functor $\lim : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$ by the universal properties of sequential colimit and limit.

Remark A.4. If we further require $\mathcal{C}$ to be presentably symmetric monoidal and equip $\mathrm{Fil}(\mathcal{C})$ with a symmetric monoidal structure through Day convolution, then the trivial filtration functor $c : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ and the sequential colimit functor $\mathrm{colim} : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$ are symmetric monoidal. We can prove this by identifying $\mathrm{Fil}(\mathcal{C})$ with $P(\mathbb{Z}^{\leqslant}) \otimes \mathcal{C}$.

Example A.5. Given a filtered object

$$X = (\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots) \in \mathrm{Fil}(\mathcal{C}).$$

By shifting the grading, we can construct another filtered object $\Sigma^{0,-1}X$. Namely, we set

$$(\Sigma^{0,-1}X)_n := X_{n+1}$$

with the same structure maps as those in X. This functor automatically admits an inverse $\Sigma^{0,1}$ given by shifting the grading along the opposite direction.

The structure maps of X also determine a natural transformation $\overline{\tau} : \Sigma^{0,-1} \to id$ of the form

$$\begin{tikzcd}
\Sigma^{0,-1}X : \arrow[d, "\overline{\tau}"] & \cdots \arrow[r] & X_{n+2} \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & \cdots \\
X : & \cdots \arrow[r] & X_{n+1} \arrow[r] & X_n \arrow[r] & X_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Since colimits and limits in $\mathrm{Fil}(\mathcal{C})$ are computed level-wise, $\Sigma^{0,-1}$ commutes with the categorical suspension, so we can extend this to a bi-grading on $\mathrm{Fil}(\mathcal{C})$ by setting

$$\Sigma^{m,n} := \Sigma^m \circ \Sigma^{0,m+n} = \Sigma^m \circ (\Sigma^{0,1})^{m+n}$$

where $\Sigma = \Sigma^{1,-1}$ is the categorical suspension.

As desired, we may regard a filtered object in $\mathcal{C}$ as a spectral sequence through [Lura, Section 1.2.2].

Construction A.6. Let $\mathrm{Gr}(\mathcal{C})$ be the category of graded objects in $\mathcal{C}$, which is defined as the category of $\mathcal{C}$-valued presheaves on the discrete category $\mathbb{Z}$. Consider the associated graded functor

$$\mathrm{Gr} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$$

with $\mathrm{Gr}(X)_n := \mathrm{cofib}(X_{n+1} \to X_n)$.

Given a filtered object $X \in \mathrm{Fil}(\mathcal{C})$, recall that the cofiber sequence

$$X_{n+1} \to X_n \to \mathrm{Gr}(X)_n$$

induces a long exact sequence

$$\cdots \to \pi_m^{\heartsuit}(X_{n+1}) \to \pi_m^{\heartsuit}(X_n) \to \pi_m^{\heartsuit}(\mathrm{Gr}(X)_n) \to \pi_{m-1}^{\heartsuit}(X_{n+1}) \to \cdots$$

on t-structure homotopy group. Taking the coproduct of these long exact sequences as n varies, we get an exact couple

$$\begin{tikzcd}
\bigoplus_{m,n\in\mathbb{Z}} \pi_m^{\heartsuit}(X_{n+1}) \arrow[rr] & & \bigoplus_{m,n\in\mathbb{Z}} \pi_m^{\heartsuit}(X_n) \arrow[dl] \\
& \bigoplus_{m,n\in\mathbb{Z}} \pi_m^{\heartsuit}(\mathrm{Gr}(X)_n) \arrow[ul] &
\end{tikzcd}$$

which further induces a spectral sequence. We denote the corresponding spectral sequence by $E(X) = E_r^{m,n}(X)$.

Remark A.7. Although we do not require this in general, in most cases, the t-structure on $\mathcal{C}$ is compatible with filtered colimits that exist in $\mathcal{C}$. In particular, the sequential colimit is usually t-exact, but the sequential limit is not in general.

To avoid taking the sequential limit, we may expect to consider only the following full subcategory of filtered objects.

Definition A.8. We say a filtered object $X \in \mathrm{Fil}(\mathcal{C})$ is τ-complete, or simply complete, if the sequential limit $\lim_n X_n$ is trivial. Let $\widehat{\mathrm{Fil}}(\mathcal{C})$ be the full subcategory spanned by the τ-complete filtrations. Then the inclusion $\widehat{\mathrm{Fil}}(\mathcal{C}) \hookrightarrow \mathrm{Fil}(\mathcal{C})$ admits a left adjoint

$$(-)^{\wedge}_{\tau} : \mathrm{Fil}(\mathcal{C}) \to \widehat{\mathrm{Fil}}(\mathcal{C}).$$

computed by

$$X^{\wedge}_{\tau} \simeq \lim_m \mathrm{cofib}(\Sigma^{0,-m} X \xrightarrow{\tau^m} X)$$

level-wisely, we have

$$(X^{\wedge}_{\tau})_n \simeq \mathrm{cofib}(\lim_m X_m \to X_n).$$

We call the left adjoint $(-)^{\wedge}_{\tau}$ the τ-completion functor, or simply completion.

When restricted to τ-complete filtrations, we have the following result.

Proposition A.9. *The associated graded functor*

$$\mathrm{Gr} : \widehat{\mathrm{Fil}}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$$

is conservative when restricted to τ-complete filtrations.

Proof. Given a morphism $f : X \to Y \in \widehat{\mathrm{Fil}}(\mathcal{C})$, to show that f is an equivalence, it suffices to show

$$\mathrm{cofib}(\Sigma^{0,-m} X \xrightarrow{\tau^m} X) \xrightarrow{f} \mathrm{cofib}(\Sigma^{0,-m} Y \xrightarrow{\tau^m} Y)$$

is an equivalence for all m by completeness. Moreover, by induction along the cofiber sequence

$$\Sigma^{0,-1}\mathrm{cofib}(\Sigma^{0,-m} X \xrightarrow{\tau^m} X) \to \mathrm{cofib}(\Sigma^{0,-m-1} X \xrightarrow{\tau^{m+1}} X) \to \mathrm{cofib}(\Sigma^{0,-1} X \xrightarrow{\tau} X),$$

it suffices to show the equivalence for $m = 1$. This is equivalent to the assumption that $\mathrm{Gr}(f)$ is an equivalence by definition. □

Except for its associated graded object, a complete filtration $X \in \widehat{\mathrm{Fil}}(\mathcal{C})$ also captured the interaction between different associated graded pieces. To be more precise, in [Ari21], Ariotta proves the equivalence between $\widehat{\mathrm{Fil}}(\mathcal{C})$ and the ∞-category of coherent cochain complexes in $\mathcal{C}$.

Following [Joy08], we make the following definition.

Definition A.10. Let Ξ be a pointed 1-category with set objects $\mathbb{Z}_*$, the set of integers with an extra base point $*$ which is a zero object; that is, both an initial and final object. For any two integers $m, n \in \mathbb{Z}$, the hom-set is defined as

$$\mathrm{Hom}_{\Xi}(m,n) := \begin{cases} \{*\}, & \text{if } n \neq m, m-1; \\ \{id, *\}, & \text{if } n = m; \\ \{\delta, *\}, & \text{if } n = m-1, \end{cases}$$

where $*$ is the unique morphism factor through the zero object $*$. The only undetermined composition is given by $\delta \circ \delta \simeq *$.

A coherent cochain complex in $\mathcal{C}$ is a pointed $\mathcal{C}$-valued presheaf over Ξ; that is, a functor $\Xi^{\mathrm{op}} \to \mathcal{C}$ between pointed ∞-categories. We denote the ∞-category of coherent cochain complexes in $\mathcal{C}$ by

$$\mathrm{Ch}(\mathcal{C}) \simeq \mathrm{Fun}_*(\Xi^{\mathrm{op}}, \mathcal{C}).$$

Theorem A.11. *There exists an equivalence between the category of complete filtered objects and coherent cochain complexes in* $\mathcal{C}$

$$\mathcal{A} : \widehat{\mathrm{Fil}}(\mathcal{C}) \simeq \mathrm{Ch}(\mathcal{C}).$$

This equivalence sends a complete filtration $X \in \widehat{\mathrm{Fil}}(\mathcal{C})$ *to the coherent cochain complex*

$$\cdots \xrightarrow{\delta} \Sigma^{-1}\mathrm{Gr}(X)_{-1} \xrightarrow{\delta} \mathrm{Gr}(X)_0 \xrightarrow{\delta} \Sigma^1\mathrm{Gr}(X)_1 \xrightarrow{\delta} \Sigma^2\mathrm{Gr}(X)_2 \xrightarrow{\delta} \cdots .$$

By construction, the point-wise t-structure homotopy group of $\mathcal{A}(X)$ coincides with the E_1-page of the spectral sequence $E(X)$.

Our next goal is to introduce the décalage functor

$$\mathrm{Déc} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Fil}(\mathcal{C}),$$

which relies on the Beilinson t-structure on $\mathrm{Fil}(\mathcal{C})$. The Beilinson t-structure was first defined in [Bei87] by Beilinson, and a modern ∞-categorical version can be found in [BMS19]. We now continue following [Ari21].

Definition A.12. The point-wise t-structure on $\mathrm{Ch}(\mathcal{C})$ defines a t-structure on $\widehat{\mathrm{Fil}}(\mathcal{C})$, and further determines a t-structure on $\mathrm{Fil}(\mathcal{C})$ along the localization $\mathrm{Fil}(\mathcal{C}) \to \widehat{\mathrm{Fil}}(\mathcal{C})$, which we call the Beilinson t-structure.

More precisely, a filtered object $X \in \mathrm{Fil}(\mathcal{C})$ is connective under the Beilinson t-structure if and only if the completion of X is connective in $\widehat{\mathrm{Fil}}(\mathcal{C}) \simeq \mathrm{Ch}(\mathcal{C})$; that is

$$\mathrm{Fil}(\mathcal{C})^{\mathrm{Bei}}_{\geqslant 0} = \{X \in \mathrm{Fil}(\mathcal{C}) : \Sigma^n\mathrm{Gr}(X)_n \in \mathcal{C}_{\geqslant 0} \text{ for all } n \in \mathbb{Z}\}.$$

Similarly, a filtered object $X \in \mathrm{Fil}(\mathcal{C})$ is coconnective under the Beilinson t-structure if and only if the X itself is complete and is coconnective in $\widehat{\mathrm{Fil}}(\mathcal{C}) \simeq \mathrm{Ch}(\mathcal{C})$; that is

$$\mathrm{Fil}(\mathcal{C})^{\mathrm{Bei}}_{\leqslant 0} = \{X \in \widehat{\mathrm{Fil}}(\mathcal{C}) \subset \mathrm{Fil}(\mathcal{C}) : \Sigma^n\mathrm{Gr}(X)_n \in \mathcal{C}_{\leqslant 0} \text{ for all } n \in \mathbb{Z}\}.$$

Definition A.13. The décalage functor $\mathrm{Déc} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Fil}(\mathcal{C})$ is defined to be the composite

$$\mathrm{Déc} : \mathrm{Fil}(\mathcal{C}) \xrightarrow{\tau^{\mathrm{Bei}}_{\geqslant *}} \mathrm{Fil}(\mathrm{Fil}(\mathcal{C})) \xrightarrow{\mathrm{colim}} \mathrm{Fil}(\mathcal{C}).$$

In particular, it sends $X \in \mathrm{Fil}(\mathcal{C})$ to the filtered object with n-th stage

$$(\mathrm{Déc}X)_n \simeq \mathrm{colim}\, \tau^{\mathrm{Bei}}_{\geqslant n} X,$$

the sequential colimit of the n-th Beilinson connective cover of X.

Since the Beilinson t-structure is essentially defined on the complete filtered objects, the décalage functor also interacts well with the completeness.

Proposition A.14. *Let* $f : X \to Y \in \mathrm{Fil}(\mathcal{C})$ *be a functor between filtered objects that induces an equivalence up to completion. Then its décalage*

$$\mathrm{Déc}f : \mathrm{Déc}X \to \mathrm{Déc}Y$$

is also an equivalence up to completion.

Proof. By Proposition A.9, it suffices to show that $\mathrm{Déc} f$ induces an equivalence on associated graded objects. Note that the associated graded pieces of $\mathrm{Déc} X$ is, by definition,

$$\operatorname{colim} \pi_*^{\mathrm{Bei},\heartsuit}(X) \in \mathcal{C}$$

where $\pi_*^{\mathrm{Bei},\heartsuit}(X) \in \mathrm{Fil}(\mathcal{C})^{\mathrm{Bei},\heartsuit} \subset \mathrm{Fil}(\mathcal{C})$ is understood as a filtered object in $\mathcal{C}$.

Regarded as a coherent cochain complex, $\pi_*^{\mathrm{Bei},\heartsuit}(X)$ is given by the t-structure homotopy group of

$$\cdots \xrightarrow{\delta} \Sigma^{-1}\mathrm{Gr}(X)_{-1} \xrightarrow{\delta} Gr(X)_0 \xrightarrow{\delta} \Sigma^1\mathrm{Gr}(X)_1 \xrightarrow{\delta} \Sigma^2\mathrm{Gr}(X)_2 \xrightarrow{\delta} \cdots,$$

which only depends on associated graded pieces of X. Then the statement follows. $\square$

If we further require the t-structure on $\mathcal{C}$ to be well-behaved, we get the following properties of the Beilinson t-structure and the décalage functor.

Proposition A.15. *(1) If the t-structure on $\mathcal{C}$ is compatible with filtered colimits (that exist in $\mathcal{C}$), the Beilinson t-structure is also compatible with filtered colimits, so the décalage preserves filtered colimits;*

(2) Suppose $\mathcal{C}$ is presentably symmetric monoidal and the t-structure on $\mathcal{C}$ is compatible with filtered colimits and the symmetric monoidal structure. Then the Beilinson t-structure is compatible with the symmetric monoidal structure given by Day convolution. In particular, the décalage functor is lax symmetric monoidal;

(3) If the t-structure on $\mathcal{C}$ is right complete, then the Beilinson t-structure is also right complete. In addition, the décalage preserves the sequential colimit functor

$$\operatorname{colim} : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}.$$

Proof. (1) The Beilinson t-structure is compatible with filtered colimits by [Ant24, Proposition 3.31]. Since colimits in $\mathrm{Fil}(\mathcal{C})$ are computed level-wise, Déc preserves filtered colimits by definition.

(2) Note that the connectivity under the Beilinson t-structure is detected by associated graded objects and the associated graded functor

$$\mathrm{Gr} : \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$$

is strongly symmetric monoidal. Then the Beilinson t-structure is compatible with the symmetric monoidal structure by definition. To be more precise, the unit of $\mathrm{Fil}(\mathcal{C})$ is the trivial filtration on the unit of $\mathcal{C}$, and its associated graded pieces are trivial except for $\mathrm{Gr}(c\mathbb{1}_{\mathcal{C}})_0 \simeq \mathbb{1}_{\mathcal{C}}$ at zero. So the unit of $\mathrm{Fil}(\mathcal{C})$ is Beilinson connective.

Moreover, let X, Y be two Beilinson connective filtered objects. Note that

$$\Sigma^n \mathrm{Gr}(X \otimes Y)_n \simeq \bigoplus_{p+q=n} \Sigma^p \mathrm{Gr}(X)_p \otimes \Sigma^q \mathrm{Gr}(Y)_q,$$

so $X \otimes Y$ is Beilinson connective because $\mathcal{C}_{\geqslant 0}$ is closed under finite coproducts and filtered colimit, and thus infinite coproducts.

Therefore, the Beilinson-Whitehead tower is lax symmetric monoidal, so Déc is lax symmetric monoidal.

(3) The right completeness of the Beilinson t-structure follows from [Ant24, Proposition 3.31]. So, for any filtered object $X \in \mathrm{Fil}(\mathcal{C})$,

$$\operatorname{colim}(\mathrm{Déc}X) \simeq \operatorname*{colim}_n \operatorname{colim} \tau^{\mathrm{Bei}}_{\geqslant n} X \simeq \operatorname{colim} X$$

by the right completeness. $\square$

The most important property of décalage is that, as desired, it preserves the associated spectral sequences up to a page-turning and re-grading. This is proved in [Ant24, Theorem 4.13].

Theorem A.16. *For $r \geqslant 1$, there are natural isomorphisms*

$$E_r^{*,*}(Dec(X)) \cong E_r^{*,*}(X)$$

up to re-grading that are compatible with the differentials on the $E_r(Dec(X))$ and $E_{r+1}(X)$-pages.

We end this section with a comparison theorem between the décalage functor defined above and the cosimplicial décalage introduced in [Lev15].

Definition A.17. Let $X_\bullet \in \mathcal{C}^\Delta$ be a cosimplicial object in $\mathcal{C}$. The cosimplicial décalage of $X_\bullet$ is defined as the filtered object

$$\operatorname{Tot} \tau_{\geqslant *} X_\bullet \simeq \lim_{\bullet \in \Delta} \tau_{\geqslant *} X_\bullet \in \mathrm{Fil}(\mathcal{C}).$$

Here, the connective cover $\tau_{\geqslant *}$ is computed level-wise. This determines the cosimplicial décalage functor

$$\mathrm{Déc}_\Delta : \mathcal{C}^\Delta \to \mathrm{Fil}(\mathcal{C}).$$

As shown in [Lev15], the spectral sequence associated to $\mathrm{Déc}_\Delta X_\bullet$ is isomorphic to the Bousfield-Kan spectral sequence of $X_\bullet$ up to a page-turning and re-grading. Note that the Bousfield-Kan spectral sequence is defined as the spectral sequence induced by the coskeleton tower on $\operatorname{Tot} X_\bullet$ through the Dold-Kan correspondence.

The Dold-Kan correspondence only produces a non-negatively graded filtration. To compare the décalage functor with the cosimplicial one, we first need to modify the Dold-Kan construction to get a complete filtration.

Construction A.18. Let $X_\bullet \in \mathcal{C}^\Delta$ be a cosimplicial object in $\mathcal{C}$. We denote the n-coskeleton of $X_\bullet$ by

$$\operatorname{Tot}_n X_\bullet := \lim_{\bullet \in \Delta^{\leqslant n}} X_\bullet \in \mathcal{C}$$

and set $\operatorname{Tot}_n X_\bullet \simeq 0$ for $n < 0$. We expect to consider the completion of the coskeleton tower

$$\cdots \to \operatorname{Tot}_n X_\bullet \to \operatorname{Tot}_{n-1} X_\bullet \to \cdots \to \operatorname{Tot}_0 X_\bullet \to \operatorname{Tot}_{-1} X_\bullet \simeq 0 \to 0 \to \cdots.$$

More precisely, let

$$\operatorname{Tot}^n X_\bullet := \operatorname{fib}(\operatorname{Tot} X_\bullet \to \operatorname{Tot}_n X_\bullet).$$

Then, since fiber commutes with sequential limit, we have

$$\lim_n \operatorname{Tot}^n X_\bullet \simeq \operatorname{fib}(\operatorname{Tot} X_\bullet \to \lim_n \operatorname{Tot}_n X_\bullet) \simeq 0.$$

Consider the tower of cofiber sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Tot}^n X_\bullet \arrow[r] \arrow[d] & \operatorname{Tot}^{n-1} X_\bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \operatorname{Tot}^0 X_\bullet \arrow[r] \arrow[d] & \operatorname{Tot}^{-1} X_\bullet \simeq \operatorname{Tot} X_\bullet \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \operatorname{Tot} X_\bullet \arrow[r] \arrow[d] & \operatorname{Tot} X_\bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \operatorname{Tot} X_\bullet \arrow[r] \arrow[d] & \operatorname{Tot} X_\bullet \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \operatorname{Tot}_n X_\bullet \arrow[r] & \operatorname{Tot}_{n-1} X_\bullet \arrow[r] & \cdots \arrow[r] & \operatorname{Tot}_0 X_\bullet \arrow[r] & \operatorname{Tot}_{-1} X_\bullet \simeq 0 \arrow[r] & \cdots
\end{tikzcd}$$

Note that after completing each horizontal filtered object, we still get a cofiber sequence of filtered objects. Because the completion of the constant filtration on $\operatorname{Tot} X_\bullet$ is trivial, the completion of the top row and the bottom row coincide up to a suspension.

For some technical reason, we define the modified Dold-Kan construction

$$\mathrm{mDK}(X_\bullet) \in \mathrm{Fil}(\mathcal{C})$$

of $X_\bullet$ as a degree shift of the top row. Namely,

$$\mathrm{mDK}(X_\bullet)_n := \operatorname{Tot}^{n-1} X_\bullet \simeq \mathrm{fib}(\operatorname{Tot} X_\bullet \to \operatorname{Tot}_{n-1} X_\bullet).$$

Theorem A.19. *The cosimplicial décalage of coincides with the décalage of the modified Dold-Kan construction. That is, we have natural equivalences*

$$\mathrm{Déc}_\Delta(X_\bullet) \simeq \mathrm{Déc}\ \mathrm{mDK}(X_\bullet).$$

Proof. By construction, the modified Dold-Kan construction of $X_\bullet$ is always complete. Note that the Beilinson t-structure is essentially defined on the category of complete filtered objects. Then it suffices to show the commutativity of the following diagram.

$$\begin{tikzcd}
\mathcal{C}^\Delta \arrow[rr, "\mathrm{mDK}"] \arrow[d, "\tau_{\geqslant *}"] & & \widehat{\mathrm{Fil}}(\mathcal{C}) \arrow[r, "\mathcal{A}", "\simeq"'] \arrow[d, "\tau^{\mathrm{Bei}}_{\geqslant *}"] & \mathrm{Ch}(\mathcal{C}) \arrow[d, "\tau_{\geqslant *}"] \\
\mathrm{Fil}(\mathcal{C}^\Delta) \arrow[rr, "\mathrm{mDK}"] \arrow[dr, "\mathrm{Tot}"'] & & \mathrm{Fil}(\widehat{\mathrm{Fil}}(\mathcal{C})) \arrow[r, "\mathcal{A}", "\simeq"'] \arrow[dl, "\mathrm{colim}"] & \mathrm{Fil}(\mathrm{Ch}(\mathcal{C})) \\
& \mathrm{Fil}(\mathcal{C}) & &
\end{tikzcd}$$

The upper right square commutes by the definition of the Beilinson t-structure. For the bottom triangle, we can check this level-wisely. Indeed, for any $X \in \mathcal{C}^\Delta$, we have

$$\begin{aligned}
\operatorname{colim} \mathrm{mDK} X_\bullet &\simeq \operatorname{colim} \mathrm{fib}(\operatorname{Tot} X_\bullet \to \operatorname{Tot}_{n-1} X_\bullet) \\
&\simeq \mathrm{fib}(\operatorname{Tot} X_\bullet \to \operatorname{colim} \operatorname{Tot}_{n-1} X_\bullet) \\
&\simeq \mathrm{fib}(\operatorname{Tot} X_\bullet \to 0) \\
&\simeq \operatorname{Tot} X_\bullet.
\end{aligned}$$

It remains to check the commutativity of the upper left square. To see this, it suffices to show that the composite

$$\mathcal{C}^\Delta \xrightarrow{\mathrm{mDK}} \widehat{\mathrm{Fil}}(\mathcal{C}) \underset{\simeq}{\xrightarrow{\mathcal{A}}} \mathrm{Ch}(\mathcal{C})$$

is t-exact with respect to the level-wise t-structure.

Given any $X_\bullet \in \mathcal{C}^\Delta$, by definition, the n-th level of $\mathcal{A}(\mathrm{mDK}(X_\bullet))$ is

$$\begin{aligned}&\Sigma^n \mathrm{Gr}(\mathrm{mDK}(X_\bullet))_n\\ \simeq&\Sigma^n \mathrm{cofib}(\mathrm{fib}(\mathrm{Tot}\, X_\bullet \to \mathrm{Tot}_n\, X_\bullet) \to \mathrm{fib}(\mathrm{Tot}\, X_\bullet \to \mathrm{Tot}_{n-1}\, X_\bullet))\\ \simeq&\Sigma^n\Sigma^{-1}\mathrm{cofib}(\mathrm{Tot}_n\, X_\bullet \to \mathrm{Tot}_{n-1}\, X_\bullet)\\ \simeq&\Sigma^n \mathrm{fib}(\mathrm{Tot}_n\, X_\bullet \to \mathrm{Tot}_{n-1}\, X_\bullet).\end{aligned}$$

Recall that in the proof of the ∞-Dold-Kan correspondence, we know that

$$\mathrm{fib}(\mathrm{Tot}_n\, X_\bullet \to \mathrm{Tot}_{n-1}\, X_\bullet)$$

is a retract of $\Sigma^{-n}X_n$, as a dual statement of [Lura, Remark 1.2.4.3]. Therefore, the n-th level of $\mathcal{A}(\mathrm{mDK}(X_\bullet))$ is a retract of X_n. Since both $\mathcal{C}_{\geqslant 0}$ and $\mathcal{C}_{\leqslant 0}$ are closed under retract, the theorem follows. □

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